Ordinal Exponentiation in Homotopy Type Theory

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***Abstract*—We present two seemingly different definitions of constructive ordinal exponentiation, where an ordinal is taken to be a transitive, extensional, and wellfounded order on a set. The first definition is abstract, uses suprema of ordinals, and is solely motivated by the expected equations. The second is more concrete, based on decreasing lists, and can be seen as a constructive version of a classical construction by Sierpiński based on functions with finite support. We show that our two approaches are equivalent (whenever it makes sense to ask the question), and use this equivalence to prove algebraic laws and decidability properties of the exponential. Our work takes place in the framework of homotopy type theory, and all results are formalized in the proof assistant Agda.**

I. Introduction

In classical mathematics and set theory, ordinals have rich and interesting structure. How much of this structure can be developed in a constructive setting, such as homotopy type theory? This is not merely a question of mathematical curiosity, as classical ordinals have powerful applications as tools for establishing consistency of logical theories [1], proving termination of processes [2], and justifying induction and recursion [3], [4], which would all be valuable to have available in constructive mathematics and proof assistants based on constructive type theory. There are many constructive approaches to ordinals, such as ordinal notation systems [5], Brouwer trees [6], or wellfounded trees with finite or countable branchings [7], [8], to name a few. In this paper, we follow the Homotopy Type Theory Book [9] and consider ordinals as order types of well ordered sets, i.e., an ordinal is a type equipped with an order relation having certain properties.

Ordinals have an arithmetic theory that generalizes the one of the natural numbers. Classically, arithmetic operations are defined by case distinction and transfinite recursion. For addition and multiplication, we have:

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \lambda &= \mathsf{sup}_{\beta < \lambda}\, (\alpha + \beta) \qquad &&\text{(if } \lambda \text{ is a limit)} \end{aligned}$$

$$\begin{aligned} \alpha \times 0 &= 0 \\ \alpha \times (\beta + 1) &= (\alpha \times \beta) + \alpha \\ \alpha \times \lambda &= \mathsf{sup}_{\beta < \lambda}\, (\alpha \times \beta) \qquad &&\text{(if } \lambda \text{ is a limit)} \end{aligned}$$

Similarly, exponentiation is classically defined by:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \times \alpha \\ \alpha^\lambda &= \mathsf{sup}_{\beta<\lambda}\, \alpha^\beta \quad &&\text{(if } \lambda \text{ is a limit, } \alpha \neq 0) \\ 0^\beta &= 0 &&\text{(if } \beta \neq 0) \end{aligned} \tag{†}$$

We can recognize the equations from arithmetic on the natural numbers in the cases of 0 and successors $\beta+1$, extended by a case for limit ordinals, which ensures that each operation is continuous in the second argument. However, while the situation might seem familiar, it is in fact constructively very different: for general well orders, such a case distinction on whether an ordinal is zero, a successor, or a limit, is available if and only if the law of excluded middle holds. Constructively, the above equations are thus not a valid definition of the arithmetic operations on ordinals. However, they are still a reasonable *specification* of how the operations should behave.

For addition and multiplication, there are well known alternative constructions based on a clear visual intuition, using disjoint unions and Cartesian products of order types, respectively (cf. the related work paragraph below and especially Escardó's work [10]). These are well behaved constructively — for example, one can prove that they satisfy the specification on the left. On the other hand, subtraction (and, similarly, division) is inherently non-constructive for the approach of ordinals as well ordered sets [10, Ordinals.AdditionProperties]. For exponentiation, most classical textbooks on ordinals [11], [12], [13], [14], [15], [16] simply employ the non-constructive definition by cases. Notable exceptions are the classical monograph by Sierpiński [17] and the constructive work by Grayson [18], [19], which we will return to shortly. In fact, going back all the way to Cantor's original writings on ordinals [20], addition and multiplication are defined "explicitly" in §14, whereas exponentiation (with base $\alpha > 1$) is defined by case distinction and transfinite recursion in §18.

Perhaps one reason why constructive ordinal exponentiation has so far been underdeveloped is that the operation is somewhat non-intuitive, even in a classical setting. For example, for the first infinite ordinal $\omega = \mathsf{sup}_{n:\mathbb{N}}\, n$, continuity in the exponent (cf. the specification (†)) implies that $2^\omega = \omega$ — an equation very different from what one might expect from, for example, cardinal exponentiation. This in turn rules out an understanding of ordinal exponentiation in terms of category theoretic exponentials: even though the collection of ordinals forms a posetal category where the morphisms are functions with additional structure, exponentiation is not right adjoint to multiplication. Even worse, as we will prove in Proposition 4, any exponentiation operation which is continuous and fully satisfies the specification (†) is inherently non-constructive: such an operation can be shown to exist if and only if the law of excluded middle holds. However, we will show that we can still define exponentiation α^β whenever $\alpha \geq 1$.

We present two approaches to ordinal exponentiation in homotopy type theory:

- An abstract construction based on suprema of ordinals and transfinite recursion, which can be proven to satisfy the specification (†) when $\alpha \geq 1$, i.e., when α has a least element (Theorem 9). This construction makes use of set quotients and the univalence axiom.
- A concrete construction via decreasing lists, inspired by Sierpiński's [17] classical construction. We show that it satisfies the specification (†) when α has a *trichotomous* least element, i.e., an element $\bot : \alpha$ such that either $x = \bot$ or $\bot < x$ for every element $x : \alpha$ (Theorem 20). This definition works in plain Martin-Löf type theory, but still requires the univalence axiom to prove many properties about it.

These two approaches have different advantages. For example, the abstract construction allows convenient proofs of algebraic laws such as $\alpha^{\beta+\gamma} = \alpha^\beta \times \alpha^\gamma$ using the universal property of the supremum as a least upper bound, while the concrete definition using decreasing lists is easily seen to preserve properties such as trichotomy or decidable equality.

Our main result is that the two approaches are actually equivalent (Theorem 24). Thanks to univalence (representation independence), we can transport properties along this equivalence and get the best of both worlds. We thus show that ordinal exponentiation can be defined in homotopy type theory in a way that works very well, even constructively; and this may be seen as an example where univalence helps for work that happens purely on the level of sets.

In a classical setting, where the specification (†) fully defines exponentiation, our definitions are necessarily equivalent to the usual definition by case distinction, and everything we do works for this standard definition. Therefore, our development is a generalization of, rather than an alternative to, the existing theory of classical ordinal exponentiation.

In this paper, we heavily use the technique of calculating with *initial segments*. This approach to working with ordinals is perhaps underused, and we aim to demonstrate its effectiveness. It is based on the fact that an ordinal (or element of an ordinal) is fully characterized by what its predecessors are — a property known as *extensionality*. The predecessors of an ordinal α are, by definition, the initial segments of α. Therefore, whenever we suggest a new construction with ordinals, we also characterize the initial segments of the resulting ordinal (cf. Propositions 7 and 19). The approach is reminiscent of how, in homotopy type theory, one often characterizes the path spaces of types that one needs in constructions.

Related Work

Ordinals in homotopy type theory: In the context of homotopy type theory, the definition of ordinals as transitive, extensional, wellfounded orders was suggested in the Homotopy Type Theory Book [9]. Escardó [10] developed a substantial Agda formalization of ordinal theory with many new results, on which our formalization is based. We are also building on our own previous work [21], where we gave implementations of addition and multiplication for the ordinals we consider in this paper, but notably no implementation of exponentiation.

In our previous work [21], we further compared the notion of ordinals considered here with other notions of constructive ordinals, namely Cantor Normal Forms [20], [22] and Brouwer trees [23], [24]. In a classical setting, these are simply different representations for ordinals, and one can (as long as it makes sense size-wise) convert between them. Constructively, the different notions split apart as there is a trade-off between decidability and the ability to calculate unrestricted limits or suprema. While Cantor Normal Forms enjoy excellent decidability properties, they only allow the calculation of finite limits or suprema. For Brouwer trees, some properties are decidable, but only very specific infinite limits can be calculated. The ordinals considered in this paper enjoy no decidable properties but allow the formation of arbitrary (small) suprema. They are also the most general ones, in the sense that Cantor Normal Forms and Brouwer trees (as well as other notions of ordinals) can be viewed as a subtype of the type of transitive extensional wellfounded orders.

Exponentiation as lists: In a classical setting, the realization of ordinal exponentiation as finitely supported functions is well known. Further, in such a setting, the implementation of exponentiation ω^β with base ω as decreasing lists of ordered pairs (n, b) is also known to proof theorists working on ordinal analysis. For example, we first came across this idea in Setzer's survey article [25]. Hancock [26] discusses the cases of 2^β and ω^β, but admits that the definition in terms of finite support is "rather non-constructive, and admittedly rather hard to motivate".

A construction of ordinal exponentiation as decreasing lists for an arbitrary base α was suggested by Grayson in his PhD thesis [18], with the relevant part published as Grayson [19]. Compared to our construction, Grayson's does not include the condition that the base has a trichotomous least element. Grayson's construction, which comes without any proofs, is thus significantly more general — but also, unfortunately, incorrect in this generality. The special case that works is equivalent to our suggestion (cf. Section VI). Grayson further claims that his construction satisfies a "recursion equation", which cannot hold in the generality claimed. However, when read as a recursive *definition*, it yields precisely our abstract construction with the caveat that Grayson uses setoids which we avoid thanks to univalence.

Mechanization of ordinal exponentiation: As far as we know, this is the first paper that comes with a mechanization of constructive ordinal exponentiation in a proof assistant. However, others have done so in a classical logic, where the definition is considerably more straightforward. For example, the Lean mathematical library [27] defines ordinal exponentiation by the case distinction (†), while Blanchette, Popescu and Traytel [28] use classical logic to encode exponentials as functions with finite support in Isabelle/HOL. However, we emphasize that we do not give a different, or "yet another" definition in this paper — in a classical setting, it is not hard to show that all these definitions coincide, including ours. Rather, we give a definition that improves on the existing ones, in the sense that it is well behaved in the absence of classical logic and (depending on the setting) may be better from the point of view of computation.

Outline and Contributions

We start by recalling how ordinals are defined in homotopy type theory in Section II, where we also discuss their basic properties and introduce a precise specification of exponentiation. In Section III, we give a first constructive implementation of exponentiation α^β using an abstract approach via suprema, and prove it well behaved when the base is positive, i.e., when $\alpha \geq 1$ (Theorem 9). In Section IV, we give a second constructive implementation via decreasing lists, which is more concrete. We show that it is well behaved when the base has a trichotomous least element (Theorem 20). In Section V, we then compare the two approaches: in Theorem 24, we show that they are in fact equivalent when the base has a trichotomous least element (and thus in particular is positive). Further, we explain how the concrete implementation in the form of decreasing lists can be seen as a type of normal forms for the abstract implementation (Theorem 30). Section VI discusses Grayson's suggestion of ordinal exponentiation via decreasing lists and the connection with our work. Finally, in Section VII, we explore what is not possible to achieve constructively for ordinal exponentiation, by giving several properties that hold in the classical theory of ordinals, but are in fact all constructively equivalent to the law of excluded middle.

Formalization

All our results have been formalized in the Agda proof assistant, and type check using Agda 2.7.0.1. Our formalization [29] is building on, and part of the TypeTopology development [10] by Escardó and contributors. Our Agda code is available on GitHub at https://github.com/martinescardo/TypeTopology and archived on Zenodo as doi:10.5281/zenodo.15461750. A browseable HTML rendering of all Agda verified results in this paper is available at https://www.cs.bham.ac.uk/~mhe/TypeTopology/Ordinals.Exponentiation.Paper.html. Throughout the paper, the symbol ⚙ is a clickable link to the corresponding machine-checked statement in that HTML file.

We found Agda extremely valuable when developing the decreasing list approach to exponentiation, as its intensional nature makes for rather combinatorial arguments that we found challenging to rigorously check on paper.

Setting and Notation

We work in intensional Martin-Löf type theory extended with the univalence axiom and set quotients. We also use function extensionality tacitly, as it follows from the univalence axiom. In particular, everything we do works in homotopy type theory as introduced in the "HoTT book" [9]. Some of the constructions work in a more minimalistic setting, and our Agda formalization (but not this paper) tracks assumptions explicitly. The univalence axiom is not only used to conveniently transport properties between equivalent representations, but is also crucial for proving equations of ordinals.

Regarding notation, we follow the aforementioned book. In particular, the identity type is denoted by $a = b$, while definitional (a.k.a. judgmental) equality is written $a \equiv b$. In the paper, we keep universe levels implicit. Recall that a type A is called a *proposition* if it has at most one inhabitant, i.e., if $x = y$ holds for all $x, y : A$. A type A is called a *set* if all its identity types $a = b$ for $a, b : A$ are propositions. We write LEM for the law of excluded middle, which claims that P or not P holds for every proposition P, i.e., LEM $:\equiv \forall(P : \mathsf{Prop}).P + \neg P$. Since we work constructively, we do not assume LEM, but will explicitly flag its appearance. Our main use of LEM is to show that other assumptions imply it, and thus have no chance of being constructively provable — they are constructive taboos.

II. Ordinals in Homotopy Type Theory

We recall that an *ordinal* in homotopy type theory is a type α equipped with a proposition-valued binary relation $<$, called the (strict) order, that is transitive, extensional and wellfounded [9, §10.3]. *Extensionality* means that two elements $x, y : \alpha$ are equal if and only if they have the same predecessors, i.e., $x = y$ if and only if $z < x \iff z < y$ holds for all $z : \alpha$. As observed by Escardó [10, Ordinals.Type], this implies that the underlying type of an ordinal is a set. *Wellfoundedness* is defined via an accessibility predicate, but is equivalent to transfinite induction: given a type family P over α, to prove $P(x)$ for all $x : \alpha$, it suffices to prove $P(x)$ for all $x : \alpha$ assuming that $P(y)$ already holds for $y < x$, i.e.,

$$\forall(x : \alpha).\big(\forall(y : \alpha).y < x \to P(y)\big) \to P(x)$$

implies $\forall(x : \alpha).P(x)$. Simple examples of ordinals include the finite types $\mathbf{n} = \{0 < 1 < \ldots < n-1\}$, and the infinite ordinal ω with underlying type $\mathbb{N}$ and order relation given by the usual order on $\mathbb{N}$. Any proposition P can be viewed as an ordinal in a trivial way, where the order is chosen to be constantly empty. Moreover, the type Ω of truth values, i.e. the subtype of the universe containing exactly the propositions, is an ordinal, where $P < Q$ is defined to mean $\neg P \times Q$.

Given an element a of an ordinal α, the *initial segment* $\alpha \downarrow a$ determined by a is the ordinal with underlying type

$$\alpha \downarrow a :\equiv \Sigma(x : \alpha).\, x < a$$

and with order relation inherited from α [9, after Ex 10.3.18]. This basic construction will prove absolutely fundamental in this paper, and we will often work with ordinals by characterizing their initial segments, see Eqs. (1) to (3) and Propositions 7 and 19.

A. The Ordinal of Ordinals

An *ordinal equivalence* from α to β is given by an equivalence $f : \alpha \to \beta$ between the underlying types that is order preserving, and whose inverse $f^{-1} : \beta \to \alpha$ is also order preserving (a property that follows classically, but not constructively). We note that any ordinal equivalence is also order reflecting. Thanks to univalence, the identity type $\alpha = \beta$ is equivalent to the type of ordinal equivalences from α to β [10, Ordinals.Equivalence], allowing us to construct identifications between ordinals via equivalences.

The type Ord of small ordinals is itself a (large) ordinal [9, Thm. 10.3.20] by setting

$$\alpha < \beta :\equiv \Sigma(b : \beta).\, \alpha = \beta \downarrow b,$$

i.e., α is strictly smaller than β if α is an initial segment of β determined by some (necessarily unique) element $b : \beta$; it is worth noting that this generalizes the order on Ω discussed above. We also note that proving extensionality for this order uses univalence.

Moreover, Ord forms a poset by defining $\alpha \leq \beta$ as any of the following equivalent conditions [10] (see also [30, Prop. 9]):

(i) $\gamma < \alpha$ implies $\gamma < \beta$ for all small ordinals γ;
(ii) for every $a : \alpha$, there exists a (necessarily unique) $b : \beta$ with $\alpha \downarrow a = \beta \downarrow b$;
(iii) there is a simulation $f : \alpha \to \beta$,

where a *simulation* is an order preserving function such that for all $x : \alpha$ and $y < f(x)$ we have $x' < x$ with $f(x') = y$. Note that $\leq$ is proposition-valued (in particular, (iii) is a proposition: there is at most one simulation between any two ordinals). Moreover, the relation is antisymmetric: if $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$, a fact that we will often use tacitly. The equivalence between (ii) and (iii) is due to the first fact in the following lemma.

Lemma 1 ([10, Ordinals.Maps] ⚙**).**
(i) Simulations preserve initial segments: if $f : \alpha \to \beta$ is a simulation, then $\beta \downarrow f\, a \;=\; \alpha \downarrow a$.
(ii) Simulations are injective and order reflecting.
(iii) Surjective simulations are precisely ordinal equivalences.

In a classical metatheory, every order preserving function induces a simulation, so that classically $\alpha \leq \beta$ holds if and only if there exists an order preserving function $\alpha \to \beta$. However, as we will see in Section VII, this is not true constructively. Compared to mere order preserving maps, simulations between ordinals are rather well behaved (as witnessed by Lemma 1).

An element $\bot$ of an ordinal α is *least* if $\bot \preceq a$ for all a in α, where $x \preceq y$ holds if $u < x$ implies $u < y$ for all $u : \alpha$. Note that for the ordinal of ordinals Ord, the order $\alpha \preceq \beta$ coincides with the order $\alpha \leq \beta$ via characterization (i) above. We remark that $\bot$ is the least element of α if and only if there are no elements x in α with $x < \bot$, equivalently if and only if the map $\star \mapsto \bot : \mathbf{1} \to \alpha$ is a simulation.

B. Addition and Multiplication

As is well known (cf. [10, Ordinals.Arithmetic] and [21, Thm. 61]), addition of two ordinals is given by taking the coproduct of the underlying types, keeping the original order in each component, and additionally requiring that all elements in the left component are smaller than anything in the right component. Initial segments of a sum of ordinals can be calculated as follows:

$$\begin{aligned}(\alpha + \beta) \downarrow \mathsf{inl}\, a \;&=\; \alpha \downarrow a;\\ (\alpha + \beta) \downarrow \mathsf{inr}\, b \;&=\; \alpha + (\beta \downarrow b).\end{aligned} \tag{⚙ 1}$$

Multiplication of two ordinals is given by equipping the Cartesian product of the underlying types with the reverse lexicographic order, i.e., $(a', b') < (a, b)$ holds if either $b' < b$, or $b' = b$ and $a' < a$. Initial segments of products of ordinals can be calculated as follows:

$$(\alpha \times \beta) \downarrow (a, b) \;=\; \alpha \times (\beta \downarrow b) + (\alpha \downarrow a). \tag{⚙ 2}$$

If $b' < b$, the ordinal equivalence witnessing (2) sends (a', b') to $\mathsf{inl}(a', b')$, and if $b' = b$ and $a' < a$, then (a', b') is sent to $\mathsf{inr}\, a'$. Note that neither addition nor multiplication are commutative, since e.g., $\omega + \mathbf{1} \neq \mathbf{1} + \omega$ and $\omega \times \mathbf{2} \neq \mathbf{2} \times \omega$.

C. Suprema

The poset of (small) ordinals has suprema (least upper bounds) of arbitrary families indexed by small types [31, Thm. 5.8]. Given a family of ordinals $F_\bullet : I \to \mathsf{Ord}$, its supremum $\sup F_\bullet$ can be constructed as the total space $\Sigma(i : I).\, F_i$, quotiented by the relation that identifies (i, x) and (j, y) if $F_i \downarrow x = F_j \downarrow y$. We thus have a simulation $[i, -] : F_i \leq \sup F_\bullet$ for each $i : I$. Moreover, these maps are jointly surjective and can be used to characterize initial segments of the supremum [30, Lem. 15]: for every $y : \sup F_\bullet$ there exists some $i : I$ and $x : F_i$ such that

$$y = [i, x] \text{ and } (\sup F_\bullet) \downarrow y = F_i \downarrow x. \tag{⚙ 3}$$

We stress that the existence of the pair (i, x) is expressed using $\exists$ (i.e. the propositional truncation of Σ). We will only use Eq. (3) to prove propositions so that we can use the universal property of the truncation to obtain an actual pair.

We write $\alpha \vee \beta$ for the binary join of α and β, i.e., for the supremum $\alpha \vee \beta = \sup F_\bullet$ of the two-element family $F_\bullet : \mathbf{2} \to \mathsf{Ord}$ with $F_0 = \alpha$ and $F_1 = \beta$.

An operation on ordinals is *continuous* if it commutes with suprema. For example, ordinal multiplication is continuous in its right argument:

Lemma 2 (⚙**).** *Ordinal multiplication is monotone and continuous in its right argument, i.e. $\beta \leq \gamma \to \alpha \times \beta \leq \alpha \times \gamma$ and $\alpha \times \sup F_\bullet = \sup(\alpha \times F_\bullet)$.*

Proof. The first claim is easily proved, for if $f : \beta \to \gamma$ is a simulation, then so is $(a, b) : \alpha \times \beta \mapsto (a, f\, b) : \alpha \times \gamma$. For the second claim, by the least upper bound property of $\sup(\alpha \times F_\bullet)$ and the first claim, it then suffices to show that $\alpha \times \sup F_\bullet \leq \sup(\alpha \times F_\bullet)$. Given $a : \alpha$ and $y : \sup F_\bullet$, we use Eqs. (2) and (3) to get the existence of $i : I$ and $x : F_i$ with

$$\begin{aligned}(\alpha \times \sup F_\bullet) \downarrow (a, y) &= \alpha \times (\sup F_\bullet \downarrow y) + (\alpha \downarrow a)\\ &= \alpha \times (F_i \downarrow x) + (\alpha \downarrow a)\\ &= \sup(\alpha \times F_\bullet) \downarrow [i, (a, x)]\end{aligned}$$

hence $\alpha \times \sup F_\bullet \leq \sup(\alpha \times F_\bullet)$. ☐

D. Expectations on Exponentiation

We can now state and make precise the specification (†) in the language of homotopy type theory. The following equations classically define ordinal exponentiation.

$$\begin{aligned} \alpha^0 &= \mathbf{1} \\ \alpha^{\beta+\mathbf{1}} &= \alpha^\beta \times \alpha \\ \alpha^{\sup_{i:I} F_i} &= \sup_{i:I}(\alpha^{F_i}) \quad \text{(if } \alpha \neq 0 \text{ and } I \text{ inhabited)} \\ \mathbf{0}^\beta &= \mathbf{0} \quad \text{(if } \beta \neq \mathbf{0}) \end{aligned} \tag{‡}$$

The final clause $\mathbf{0}^\beta = \mathbf{0}$ has the side condition $\beta \neq \mathbf{0}$ in order to not clash with $\alpha^0 = \mathbf{1}$. Classically, the supremum clause is equivalent to the usual equation involving limit ordinals, but formulating it for arbitrary inhabited suprema has the advantage of ensuring that exponentiation is directly continuous in the exponent. Furthermore, in this clause we ask for I to be inhabited (i.e., we have an element of its propositional truncation) rather than the classically equivalent requirement that I is nonempty, since we will use the specification in a constructive setting.

Lemma 3 (⚙). *The zero and successor clauses in the specification together imply the equations $\alpha^{\mathbf{1}} = \alpha$ and $\alpha^{\mathbf{2}} = \alpha \times \alpha$. Moreover, the supremum clause in the specification implies that $\alpha^{(-)}$ is monotone for $\alpha \neq 0$, i.e., if $\beta \leq \gamma$ then $\alpha^\beta \leq \alpha^\gamma$ for $\alpha \neq 0$.* ☐

The bad news is that constructively there cannot be an operation satisfying all of the equations in (‡):

Proposition 4 (⚙). *There is an exponentiation operation*

$$\exp : \mathsf{Ord} \times \mathsf{Ord} \to \mathsf{Ord}$$

satisfying the specification (‡) if and only if LEM *holds.*

Proof. Using LEM, such an operation can be defined by cases. Conversely, suppose we had such an operation exp and let P be an arbitrary proposition. As explained in the beginning of the section, any proposition can be viewed as an ordinal, and we consider the sum of two such ordinals: $\alpha :\equiv P + \mathbf{1}$. Obviously $\alpha \neq \mathbf{0}$ and $\mathbf{0} \leq \mathbf{1}$, so Lemma 3 yields $\mathbf{1} = \exp(\alpha, \mathbf{0}) \leq \exp(\alpha, \mathbf{1}) = \alpha$. Hence, we get a simulation $f : \mathbf{1} \to \alpha$. Now, either $f\star = \mathsf{inl}\, p$, in which case P holds, or $f\star = \mathsf{inr}\star$. In the latter case, we can prove $\neg P$: simulations preserve least elements, so assuming $p : P$ we must have $f\star = \mathsf{inl}\, p$, which contradicts $f\star = \mathsf{inr}\star$. ☐

The good news is that, if we assume that the base is positive, we *can* define a well behaved ordinal exponentiation operation α^β that satisfies the specification (‡). Assuming $\alpha \geq \mathbf{1}$, it is convenient to consider a stronger specification of exponentiation which combines the zero and supremum cases:

$$\begin{aligned} \alpha^{\beta+\mathbf{1}} &= \alpha^\beta \times \alpha \\ \alpha^{\sup_{i:I} F_i} &= \mathbf{1} \vee \sup_{i:I}(\alpha^{F_i}) \end{aligned} \tag{‡′}$$

Note that in the supremum case, in contrast to the regular specification (‡), we do not include any requirement that the indexing family I is inhabited. As a consequence, still under the assumption $\alpha \geq \mathbf{1}$, the stronger specification (‡′) implies the regular specification (‡): Since $\mathbf{0}$ is the empty supremum, (‡′) implies $\alpha^{\mathbf{0}} = \mathbf{1} \vee \mathbf{0} = \mathbf{1}$. Further, for an inhabited index set I, (‡′) gives $\alpha^{\sup_{i:I} F_i} = \sup_{i:I}(\alpha^{F_i})$ because in this case we get $\sup_{i:I}(\alpha^{F_i}) \geq \mathbf{1}$ as $\mathbf{1} = \alpha^{\mathbf{0}} \leq \alpha^{F_i}$ holds for any $i : I$ by monotonicity. The converse implication (‡) $\Longrightarrow$ (‡′) for $\alpha \geq \mathbf{1}$ is true classically, but does not seem provable constructively.

III. Abstract Algebraic Exponentiation

Let us start by presenting a definition of exponentiation that is fully guided by the equations we expect or want. We know from Proposition 4 that we cannot hope to define exponentiation α^β for arbitrary α and β, so in order to avoid the case distinction on whether $\alpha = 0$ or not, let us restrict our attention to the case when $\alpha \geq \mathbf{1}$. This means that it is sufficient to define an exponentiation operation which satisfies the specification (‡′). This specification is based on the classical characterization of ordinals as either successors or limits, where successor ordinals are of the form $\beta = \gamma + \mathbf{1}$ and limit ordinals are of the form $\lambda = \sup_{\gamma<\lambda} \gamma$. This classification is not available constructively, but a weaker variant of it is, where both cases are combined: every ordinal is (constructively) the supremum of the successors of its predecessors:

Lemma 5 ([19, §2.5], [32, Ex. 5.21] ⚙). *Every ordinal β satisfies the equation $\beta = \sup_{b:\beta}(\beta \downarrow b + \mathbf{1})$.* ☐

In particular, this lemma implies that the specification (‡′) *uniquely determines* α^β for $\alpha \geq \mathbf{1}$. Indeed, we have

$$\begin{aligned} \alpha^\beta &= \alpha^{\sup_{b:\beta}(\beta\downarrow b+\mathbf{1})} && \text{(by Lemma 5)} \\ &= \mathbf{1} \vee \sup_{b:\beta}(\alpha^{\beta\downarrow b+\mathbf{1}}) && \text{(to satisfy (‡′), supremum case)} \\ &= \mathbf{1} \vee \sup_{b:\beta}(\alpha^{\beta\downarrow b} \times \alpha) && \text{(to satisfy (‡′), successor case).} \end{aligned}$$

The last expression involves $\alpha^{\beta\downarrow b}$, where the exponent $\beta \downarrow b$ is strictly smaller than β, so this suggests to define α^β by transfinite induction on β to be the ordinal $\mathbf{1} \vee \sup_{b:\beta}(\alpha^{\beta\downarrow b} \times \alpha)$. Since it will be convenient to work with a single supremum instead, we adopt the following equivalent definition.

Definition 6 (Abstract exponentiation, α^β ⚙). For a given ordinal α, we define the operation $\alpha^{(-)} : \mathsf{Ord} \to \mathsf{Ord}$ by transfinite induction as follows:

$$\alpha^\beta :\equiv \sup_{\mathbf{1}+\beta}(\mathsf{inl}\star \mapsto \mathbf{1}; \mathsf{inr}\, b \mapsto \alpha^{\beta\downarrow b} \times \alpha).$$

The definition of exponentiation α^β does not rely on α being positive, but most properties of α^β will require it. Note that exponentiation itself is always positive, i.e., $\alpha^\beta \geq \mathbf{1}$ holds by construction, making it possible to iterate well behaved exponentiation. We will write $\bot :\equiv [\mathsf{inl}\star, \star]$ for the least element of an exponential α^β. Using Eqs. (2) and (3) we get the following characterization of initial segments.

Proposition 7 (Initial segments of α^β ⚙). *For $a : \alpha$, $b : \beta$ and $e : \alpha^{\beta\downarrow b}$, we have*

$$\alpha^\beta \downarrow [\mathsf{inr}\, b, (e, a)] \;=\; \alpha^{\beta\downarrow b} \times (\alpha \downarrow a) + \alpha^{\beta\downarrow b} \downarrow e. \qquad ☐$$

We can put this characterization to work immediately, to prove that abstract exponentiation is monotone in the exponent for both the weak and the strict order of ordinals.

Proposition 8 **().** *Abstract exponentiation is monotone in the exponent: if $\beta \leq \gamma$ then $\alpha^\beta \leq \alpha^\gamma$. Furthermore, if $\alpha > \mathbf{1}$, then it moreover preserves the strict order, i.e., if $\alpha > \mathbf{1}$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$.*

Proof. Let $f : \beta \to \gamma$ be a simulation. By Lemma 1 we have

$$\begin{aligned}\alpha^\beta &\equiv \mathsf{sup}_{\mathbf{1}+\beta}(\mathsf{inl}\,\star \mapsto \mathbf{1}; \mathsf{inr}\,b \mapsto \alpha^{\beta\downarrow b} \times \alpha) \\ &= \mathsf{sup}_{\mathbf{1}+\beta}(\mathsf{inl}\,\star \mapsto \mathbf{1}; \mathsf{inr}\,b \mapsto \alpha^{\gamma\downarrow fb} \times \alpha) \\ &\leq \mathsf{sup}_{\mathbf{1}+\gamma}(\mathsf{inl}\,\star \mapsto \mathbf{1}; \mathsf{inr}\,c \mapsto \alpha^{\gamma\downarrow c} \times \alpha) \equiv \alpha^\gamma,\end{aligned}$$

where the inequality holds because the supremum gives the least upper bound.

For the second claim, suppose we have $\alpha > \mathbf{1}$ and $\beta < \gamma$, i.e., we have $\mathbf{1} = \alpha \downarrow a_1$ and $\beta = \gamma \downarrow c$ for some $a_1 : \alpha$ and $c : \gamma$. Then, using Proposition 7 and the least element $\bot$ of $\alpha^{\gamma\downarrow c}$, we calculate that

$$\begin{aligned}\alpha^\gamma \downarrow [\mathsf{inr}\,c, (\bot, a_1)] &= \alpha^{\gamma\downarrow c} \times (\alpha \downarrow a_1) + \alpha^{\gamma\downarrow c} \downarrow \bot \\ &= \alpha^{\gamma\downarrow c} \times \mathbf{1} + \mathbf{0} = \alpha^{\gamma\downarrow c} = \alpha^\beta.\end{aligned}$$

Hence, $\alpha^\beta < \alpha^\gamma$, as desired. ☐

Using monotonicity, we can now prove that abstract exponentiation is well behaved whenever the base is positive.

Theorem 9 **().** *Assuming $\alpha \geq \mathbf{1}$, abstract exponentiation α^β satisfies the specification ($\ddagger'$) (and hence also the specification ($\ddagger$)).*

Proof. For the successor clause, we want to show $\alpha^{\beta+\mathbf{1}} = \alpha^\beta \times \alpha$. First note that if $\alpha \geq \mathbf{1}$, then $\alpha^{\mathbf{1}} = \alpha$, by the definition of $\alpha^{\mathbf{1}}$ as a supremum. The result then follows from the more general statement in Proposition 10 below.[1] Next, for the supremum clause of ($\ddagger'$), we want to show $\alpha^{\mathsf{sup}\,F_\bullet} = \mathbf{1} \vee \mathsf{sup}\,\alpha^{F_\bullet}$ for a given $F : I \to \mathsf{Ord}$. Since $F_i \leq \mathsf{sup}\,F_\bullet$ for all $i : I$, we have $\mathsf{sup}\,\alpha^{F_\bullet} \leq \alpha^{\mathsf{sup}\,F_\bullet}$ via Proposition 8, and hence $\mathbf{1} \vee \mathsf{sup}\,\alpha^{F_\bullet} \leq \alpha^{\mathsf{sup}\,F_\bullet}$ since $\mathbf{1} \leq \alpha^\beta$ for any β. For the reverse inequality, it suffices to prove $\mathbf{1} \leq \mathbf{1} \vee \mathsf{sup}\,\alpha^{F_\bullet}$ and

$$\alpha^{\mathsf{sup}\,F_\bullet \downarrow y} \times \alpha \leq \mathbf{1} \vee \mathsf{sup}\,\alpha^{F_\bullet}$$

for all $y : \mathsf{sup}\,F_\bullet$. The former is immediate, and for the latter, we note that Eq. (3) implies the existence of $i : I$ and $x : F_i$ such that

$$\alpha^{\mathsf{sup}\,F_\bullet \downarrow y} \times \alpha = \alpha^{F_i \downarrow x} \times \alpha \leq \alpha^{F_i} \leq \mathbf{1} \vee \mathsf{sup}\,\alpha^{F_\bullet}.$$

Finally, since $\alpha \geq \mathbf{1}$ by assumption, ($\ddagger$) follows from ($\ddagger'$). ☐

The following two propositions establish the expected connections of exponentiation with addition and multiplication, respectively. Note that they hold even without the assumption that $\alpha \geq \mathbf{1}$.

[1] In the formalization we additionally include a direct proof which is more general in terms of universe levels.

Proposition 10 **().** *For ordinals α, β and γ, we have*

$$\alpha^{\beta+\gamma} = \alpha^\beta \times \alpha^\gamma.$$

Proof. We do transfinite induction on γ. Our first observation is that $\alpha^\beta \times \alpha^\gamma = \alpha^\beta \vee \mathsf{sup}_{c:\gamma}(\alpha^\beta \times \alpha^{\gamma\downarrow c} \times \alpha)$, which follows from the fact that multiplication is associative as well as continuous on the right (Lemma 2), noting that $\vee$ is implemented as a supremum.

Applying the induction hypothesis, we can rewrite $\alpha^\beta \times \alpha^{\gamma\downarrow c}$ to $\alpha^{\beta+\gamma\downarrow c}$, which is $\alpha^{(\beta+\gamma)\downarrow \mathsf{inr}\,c}$. The remaining goal thus is

$$\alpha^{\beta+\gamma} = \alpha^\beta \vee \mathsf{sup}_{c:\gamma}(\alpha^{(\beta+\gamma)\downarrow \mathsf{inr}\,c} \times \alpha),$$

which one gets by unfolding the definition on the left and applying antisymmetry of $\leq$. ☐

Proposition 11 **().** *For ordinals α, β and γ, iterated exponentiation can be calculated as follows:*

$$\left(\alpha^\beta\right)^\gamma = \alpha^{\beta\times\gamma}.$$

Proof. We proceed by transfinite induction on γ and use antisymmetry. Since exponentials are least upper bounds, and always positive, in one direction it suffices to prove $\left(\alpha^\beta\right)^{\gamma\downarrow c} \times \alpha^\beta \leq \alpha^{\beta\times\gamma}$ for all $c : \gamma$. To this end, notice that

$$\begin{aligned}\left(\alpha^\beta\right)^{\gamma\downarrow c} \times \alpha^\beta &= \alpha^{\beta\times\gamma\downarrow c} \times \alpha^\beta && \text{(by IH)} \\ &= \alpha^{\beta\times\gamma\downarrow c+\beta} && \text{(by Proposition 10)} \\ &= \alpha^{\beta\times(\gamma\downarrow c+\mathbf{1})} && \text{(since $\times$ distributes over $+$)} \\ &\leq \alpha^{\beta\times\gamma},\end{aligned}$$

where the final inequality holds because we have $\gamma \downarrow c + \mathbf{1} \leq \gamma$ (by Lemma 5), exponentiation is monotone in the exponent (Proposition 8), and multiplication is monotone on the right (Lemma 2).

For the other inequality, we show $\alpha^{(\beta\times\gamma)\downarrow(b,c)} \times \alpha \leq \left(\alpha^\beta\right)^\gamma$ for all $b : \beta$ and $c : \gamma$. Indeed we have

$$\begin{aligned}&\alpha^{(\beta\times\gamma)\downarrow(b,c)} \times \alpha \\ &= \alpha^{\beta\times(\gamma\downarrow c)+\beta\downarrow b} \times \alpha && \text{(by Eq. (2))} \\ &= \alpha^{\beta\times(\gamma\downarrow c)} \times \alpha^{\beta\downarrow b} \times \alpha && \text{(by Proposition 10)} \\ &= \left(\alpha^\beta\right)^{\gamma\downarrow c} \times \alpha^{\beta\downarrow b} \times \alpha && \text{(by IH)} \\ &\leq \left(\alpha^\beta\right)^{\gamma\downarrow c} \times \alpha^\beta && \text{(assoc. and monotonicity of $\times$)} \\ &\leq \left(\alpha^\beta\right)^\gamma.\end{aligned}$$

☐

While it is quite clear that addition and multiplication of ordinals preserve decidable equality, it is not obvious at all that exponentiation also preserves this property — exponentiation is defined as a supremum, which is defined as a quotient, and it is not the case that quotients preserve decidable equality. Luckily, the construction introduced in the next section will make this fact obvious, at least when α has a trichotomous least element.

IV. Decreasing Lists: a Constructive Formulation of Sierpiński's Definition

As discussed in the introduction, in a classical metatheory, there is a "non-axiomatic" construction of exponentials α^β for $\alpha \geq \mathbf{1}$, based on functions $\beta \to \alpha$ with finite support [17, §XIV.15]. Recall that $\alpha \geq \mathbf{1}$ means that α has a least element $\bot : \alpha$, and that a function $\beta \to \alpha$ has finite support if it is zero almost everywhere, i.e., if it differs from the least element $\bot$ for only finitely many inputs. Using classical logic, the set of functions $\beta \to \alpha$ with finite support can then be shown to be an ordinal.

Unfortunately, this construction depends on classical principles in several places. For example, the notion of being finite splits apart into several different constructive notions such as Bishop finiteness, subfiniteness, Kuratowski finiteness, etc. [33], [34], and different notions seem to be needed to show that functions with finite support form an ordinal $\beta \to_{\mathrm{fs}} \alpha$, that this ordinal $\beta \to_{\mathrm{fs}} \alpha$ satisfies the specification (‡), and so on.

Classically, a function with finite support is equivalently given by the finite collection of input-output pairs where the function is greater than zero, and this gives rise to a formulation that we found to be well behaved constructively. The finite collection of input-output pairs can be represented as a list in which the input components are ordered decreasingly, which ensures that the representation is unique and that each input has at most one output. In order to re-use results on ordinal multiplication, where the second component is dominant, i.e., $b_1 < b_2$ implies $(a_1, b_1) < (a_2, b_2)$, we swap the positions of inputs and outputs and consider lists of *output-input* pairs.

Definition 12 ($[\alpha, \beta]_<$ ⚙). For ordinals α and β, we write

$$[\alpha, \beta]_< :\equiv \Sigma(l : \mathsf{List}(\alpha \times \beta)).\ \mathsf{is\text{-}decreasing}(\mathsf{map}\,\pi_2\, l)$$

for the type of lists over $\alpha \times \beta$ decreasing in the β-component.

Remark 13 (⚙). Since the type expressing that a list is decreasing in the second component is a proposition, it follows that two elements of $[\alpha, \beta]_<$ are equal as soon as their underlying lists are equal. Accordingly, in denoting elements (l, p) of type $[\alpha, \beta]_<$, we will always omit the second proof component p, and simply write $l : [\alpha, \beta]_<$.

Following Sierpiński's construction, all outputs in the finite collection of input-output pairs should be greater than the least element. Therefore, we should be considering the type $[\alpha_{>\bot}, \beta]_<$ where, for α with least element $\bot$, we write

$$\alpha_{>\bot} :\equiv \Sigma(a : \alpha).\, a > \bot$$

for the set of all elements greater than the least element. In general, this subtype is not necessarily an ordinal:

Proposition 14 (⚙). LEM *holds if and only if, for all ordinals α, the subtype of positive elements $\alpha_{>\bot}$ is an ordinal.*

Proof. It is not hard to check that LEM allows one to prove that $\alpha_{>\bot}$ is an ordinal. For the converse, we assume that $\mathsf{Ord}_{>\mathbf{0}}$, the (large) subtype of ordinals strictly greater than $\mathbf{0}$, is an ordinal. To prove LEM it is enough to prove that the ordinal $\mathbf{2}$ of booleans and the ordinal Ω of truth values (cf. the example in Section II) are equal. So let us show that they have the same predecessors in $\mathsf{Ord}_{>\mathbf{0}}$, namely only the one-element ordinal $\mathbf{1}$. For $\mathbf{2}$ it is straightforward that its only predecessor in $\mathsf{Ord}_{>\mathbf{0}}$ is $\mathbf{1}$. For Ω, we note that if $\mathbf{0} < \alpha < \Omega$, then $\alpha = \Omega \downarrow Q$ for some proposition Q and further we have $\mathbf{0} < Q$, so that Q must hold, and hence $\alpha = \Omega \downarrow Q = \mathbf{1}$. □

To ensure that $\alpha_{>\bot}$ is an ordinal, and consequently $[\alpha_{>\bot}, \beta]_<$ as well, it suffices to require the least element $\bot$ to be *trichotomous*, meaning for all $x : \alpha$, either $x = \bot$ or $x > \bot$. As pointed out to us by Paul Levy, a trichotomous least element is simply the least element with respect to the "disjunctive order" $\leqslant$ defined by $x \leqslant y \iff (x < y) + (x = y)$.

Lemma 15 (⚙). *An ordinal α has a trichotomous least element if and only if $\alpha = \mathbf{1} + \alpha'$ for some (necessarily unique) ordinal α'. If this happens, then $\alpha' = \alpha_{>\bot}$.*

Proof. Assume α has a trichotomous least element $\bot$. We first want to show that in this case $\alpha_{>\bot}$ is an ordinal, and then that $\alpha = \mathbf{1} + \alpha_{>\bot}$. By trichotomy of $\bot$, we can prove that the order on $\alpha_{>\bot}$ inherited from α is extensional, and thus that $\alpha_{>\bot}$ is an ordinal, since transitivity and wellfoundedness is always retained by the inherited order. Using trichotomy again, we can define an equivalence $\alpha \to \mathbf{1} + \alpha_{>\bot}$ by mapping $x : \alpha$ to the left if $x = \bot$ and to the right if $x > \bot$.

The converse is immediate, and uniqueness follows as addition is left cancellable [10, Ordinals.AdditionProperties]. □

If α has a trichotomous least element, we thus have our candidate for a more concrete implementation of the exponential α^β; the following suggestion is similar to Grayson's [19], to which we come back in Section VI.

Definition 16 (Concrete exponentiation, $\mathsf{exp}\,(\alpha, \beta)$ ⚙). For ordinals α and β with α having a trichotomous least element, we write $\mathsf{exp}\,(\alpha, \beta)$ for $[\alpha_{>\bot}, \beta]_<$ (cf. Definition 12) and call it the *concrete exponentiation of α and β*.

Thanks to Lemma 15, we often choose to work with the more convenient $\mathsf{exp}\,(\mathbf{1} + \alpha', \beta) = [\alpha', \beta]_<$ rather than $\mathsf{exp}\,(\alpha, \beta) = [\alpha_{>\bot}, \beta]_<$ in the Agda formalization. We next prove that indeed $\mathsf{exp}\,(\alpha, \beta)$ can be given a rather natural order which makes it into an ordinal.

Proposition 17 (⚙). *For ordinals α and β with α having a trichotomous least element, the lexicographic order on lists makes $\mathsf{exp}\,(\alpha, \beta)$ into an ordinal that again has a trichotomous least element.*

Proof. By Lemma 15, $\alpha_{>\bot}$ is an ordinal, hence $\alpha_{>\bot} \times \beta$ is also an ordinal. The lexicographic order on $\mathsf{List}(\alpha_{>\bot} \times \beta)$ preserves key properties of the underlying order, including transitivity and wellfoundedness. Using structural induction on lists, we can show that the lexicographic order on the subset of lists decreasing in the second component is extensional. Consequently, $\mathsf{exp}\,(\alpha, \beta)$ is an ordinal and the empty list $[]$ is easily seen to be its least trichotomous element. □

We now wish to characterize the initial segments of the ordinal $\mathsf{exp}\,(\alpha,\beta)$. Before doing so we must introduce two instrumental functions (Eqs. (4) and (5)) and a lemma.

Lemma 18 (⚙). *Let α be an ordinal with a trichotomous least element. Any order preserving map $f:\beta\to\gamma$ induces an order preserving map $\overline{f}:\mathsf{exp}\,(\alpha,\beta)\to\mathsf{exp}\,(\alpha,\gamma)$ by applying f to the second component of each pair in the list.*

Moreover, if f is a simulation, then so is $\overline{f}$. Consequently, $\mathsf{exp}\,(\alpha,-)$ is monotone.

Proof. Note that order preservation of f ensures that the outputs of $\overline{f}$ are again decreasing in the second component, so that we have a well defined map which is easily seen to be order preserving. Now suppose that f is moreover a simulation. Since f is order reflecting and injective (Lemma 1), it follows that $\overline{f}$ is order reflecting. Therefore, it suffices to prove that if we have $l<\overline{f}\,l_1$, then there is $l_2:\mathsf{exp}\,(\alpha,\beta)$ with $\overline{f}\,l_2=l$. We do so by induction on l. The case $l=[\,]$ is easy and if l is a singleton, then we need only use that f is a simulation. So let $l=(a,c)::(a',c')::l'$ and $l_1=(a_1,b_1)::l_1'$. We proceed by case analysis on $l<\overline{f}\,l_1$ and work out the details in case $c<f\,b_1$; the other cases are dealt with similarly. Since f is a simulation we have $b_2:\beta$ such that $f\,b_2=c$. Since $c'<c<f\,b_1$ we have $((a',c')::l')<\overline{f}\,l_1$, and hence we get l_2' such that $\overline{f}\,l_2'=(a',c')::l'$ by induction hypothesis. Since $c'<c=f\,b_2$ and f is order reflecting, the list $l_2:\equiv(a,b_2)::l_2'$ is decreasing in the second component and by construction we have $\overline{f}\,l_2=l$ as desired. □

In particular, the construction in Lemma 18 gives us

$$\iota_b:\mathsf{exp}\,(\alpha,\beta\downarrow b)\leq\mathsf{exp}\,(\alpha,\beta)\,. \qquad (⚙\ 4)$$

for every $b:\beta$. In a sense, this map has an inverse: given a list $l:\mathsf{exp}\,(\alpha,\beta)$ such that each second component is below some element $b:\beta$, we can construct

$$\tau_b\,l:\mathsf{exp}\,(\alpha,\beta\downarrow b) \qquad (⚙\ 5)$$

by inserting the required inequality proofs. Moreover, the assignment $l\mapsto\tau_b\,l$ is order preserving and inverse to ι_b.

Proposition 19 (Initial segments of $\mathsf{exp}\,(\alpha,\beta)$, ⚙). *For ordinals α and β with α having a trichotomous least element, we have*

$$\begin{aligned}&\mathsf{exp}\,(\alpha,\beta)\downarrow((a,b)::l)\\&=\mathsf{exp}\,(\alpha,\beta\downarrow b)\times(\alpha\downarrow a)+\mathsf{exp}\,(\alpha,\beta\downarrow b)\downarrow\tau_b\,l.\end{aligned}$$

Similarly, for $a:\alpha_{>\bot}$, $b:\beta$ and $l:\mathsf{exp}\,(\alpha,\beta\downarrow b)$, we have

$$\begin{aligned}&\mathsf{exp}\,(\alpha,\beta)\downarrow((a,b)::\iota_b\,l)\\&=\mathsf{exp}\,(\alpha,\beta\downarrow b)\times(\alpha\downarrow a)+\mathsf{exp}\,(\alpha,\beta\downarrow b)\downarrow l.\end{aligned}$$

Proof. The second claim follows from the first and the fact that τ_b and ι_b are inverses.

For the first claim, we construct order preserving functions f and g in each direction, and show that they are inverse to each other. We define an order preserving map

$$\begin{aligned}f:\ &\mathsf{exp}\,(\alpha,\beta)\downarrow((a,b)::l)\\&\to\mathsf{exp}\,(\alpha,\beta\downarrow b)\times(\alpha\downarrow a)+\mathsf{exp}\,(\alpha,\beta\downarrow b)\downarrow\tau_b\,l\end{aligned}$$

by cases on $l_0<((a,b)::l)$ via

$$f\,l_0:\equiv\begin{cases}\mathsf{inl}([\,],\bot) & \text{if } l_0=[\,];\\ \mathsf{inl}((a',b')::\tau_b\,l_1,\bot) & \text{if } l_0=(a',b')::l_1 \text{ and } b'<b;\\ \mathsf{inl}(\tau_b\,l_1,a') & \text{if } l_0=(a',b)::l_1 \text{ and } a'<a;\\ \mathsf{inr}(\tau_b\,l_1) & \text{if } l_0=(a,b)::l_1 \text{ and } l_1<l.\end{cases}$$

In the other direction, we define g, using the fact that equality with $\bot$ is decidable, by

$$\begin{aligned}g\,(\mathsf{inl}(l_1,a')):\equiv&\begin{cases}\iota_b\,l_1 & \text{if } a'=\bot;\\ (a',b)::\iota_b\,l_1 & \text{if } a'>\bot;\end{cases}\\ g\,(\mathsf{inr}\,l_1):\equiv&\ (a,b)::\iota_b\,l_1.\end{aligned}$$

Direct calculations then verify that g is order preserving and that $f\circ g=\mathrm{id}$ and $g\circ f=\mathrm{id}$. □

The upcoming Theorem 20 and Proposition 21 can be derived from the corresponding facts about abstract exponentiation and Theorem 24 below, but for comparison we include sketches of direct proofs as well (for further details, we refer the interested reader to the formalization). Alternatively, Theorem 24 can be derived from Theorem 20 and the fact that operations satisfying the specification (‡′) are unique — the proof effort is about the same for both strategies.

Theorem 20 (⚙). *Concrete exponentiation $\mathsf{exp}\,(\alpha,\beta)$ satisfies the specification (‡′) (and hence the specification (‡)) for α with a trichotomous least element.*

Proof sketch. Again, (‡) follows from (‡′) since α is assumed to be positive. The successor case follows from the more general Proposition 21 below. For the supremum clause of (‡′), i.e., to prove $\mathsf{exp}\,(\alpha,\sup F_\bullet)=\mathbf{1}\vee\sup\mathsf{exp}\,(\alpha,F_\bullet)$, we appeal to Lemma 18 to obtain simulations

$$\sigma_i:\mathsf{exp}\,(\alpha,F_i)\leq\mathsf{exp}\,(\alpha,\sup F_\bullet)\,,$$

yielding a simulation $\sigma:\mathbf{1}\vee\sup\mathsf{exp}\,(\alpha,F_\bullet)\leq\mathsf{exp}\,(\alpha,\sup F_\bullet)$. We now show that σ is additionally a surjection, and hence an ordinal equivalence. The key observation is the following lemma, which follows by induction on lists and Eq. (3):

Claim. Given $(a,[i,x])::l$ of type $\mathsf{exp}\,(\alpha,\sup F_\bullet)$, there exists $l':\mathsf{exp}\,(\alpha,F_i)$ such that $\sigma_i((a,x)::l')=(a,[i,x])::l$.

Now, for the surjectivity of σ, let $\ell:\mathsf{exp}\,(\alpha,\sup F_\bullet)$ be arbitrary. If $\ell=[\,]$, it is the image of the unique element $*:\mathbf{1}$. If $\ell=(a,y)::l$, then by Eq. (3), $y=[i,x]$ for some i and x, and we apply the claim to get l' with $\sigma[i,(a,x)::l']=\sigma_i((a,x)::l')=\ell$. □

Proposition 21 (⚙). *For ordinals α, β and γ with α having a trichotomous least element, we have $\mathsf{exp}\,(\alpha,\beta+\gamma)=\mathsf{exp}\,(\alpha,\beta)\times\mathsf{exp}\,(\alpha,\gamma)$.*

Proof sketch. We start by defining

$$\begin{aligned}f:\mathsf{exp}\,(\alpha,\beta+\gamma)&\to\mathsf{exp}\,(\alpha,\beta)\times\mathsf{exp}\,(\alpha,\gamma)\\ [\,]&\mapsto([\,],[\,]);\\ (a,\mathsf{inl}\,b)::l&\mapsto((a,b)::\pi_1(f\,l),\pi_2(f\,l));\\ (a,\mathsf{inr}\,c)::l&\mapsto(\pi_1(f\,l),(a,c)::\pi_2(f\,l)).\end{aligned}$$

The fact that this is well defined, in particular that $\pi_2(f\,l)$ yields a list that is decreasing in the second component, follows from the observation that an element of $[\alpha, \beta + \gamma]_<$ starting with $(a, \mathsf{inl}\, b)$ cannot have any $(a', \mathsf{inr}\, c)$ entries. Put differently, $\pi_2(f((a, \mathsf{inl}\, b) :: l)) = []$ for any l. A straightforward proof by case analysis shows that f is order preserving.

In the other direction, we define

$$\begin{aligned} g : \mathsf{exp}\,(\alpha, \beta) \times \mathsf{exp}\,(\alpha, \gamma) &\to \mathsf{exp}\,(\alpha, \beta + \gamma) \\ ([], []) &\mapsto []; \\ ((a, b) :: l_1, []) &\mapsto (a, \mathsf{inl}\, b) :: g(l_1, []); \\ (l_1, (a, c) :: l_2) &\mapsto (a, \mathsf{inr}\, c) :: g(l_1, l_2). \end{aligned}$$

Then one checks directly that g is order preserving and that it is the inverse of f. □

A feature of concrete exponentiation is that it preserves decidability properties.

Proposition 22 (⚙). *Assume α has a trichotomous least element. If α and β have decidable equality, then so does* $\mathsf{exp}\,(\alpha, \beta)$.

Proof. All of $\times$, List, and taking subtypes preserve decidable equality, and $\mathsf{exp}\,(\alpha, \beta)$ is a subtype of $\mathsf{List}\,(\alpha \times \beta)$. □

We recall that an ordinal α is said to be *trichotomous* if we have $(x < y) + (x = y) + (y < x)$ for every $x, y : \alpha$.

Proposition 23 (⚙). *If α and β are trichotomous, then so is* $\mathsf{exp}\,(\alpha, \beta)$.

Proof. Proved by a straightforward induction on lists. □

V. Abstract and Concrete Exponentiation

Since both the abstract and concrete constructions of ordinal exponentiation satisfy the specification (‡′), they in fact coincide whenever the base has a trichotomous least element. We give an alternative proof based on initial segments (Theorem 24), and explain its computational content by showing how it relates to a surjective denotation function, which represents elements of the abstract exponential as lists of the concrete exponential.

A. Abstract and Concrete Exponentiation Coincide

If α has a trichotomous least element, then α in particular has a least element, i.e., $\mathbf{1} \leq \alpha$. Hence both the abstract exponentiation α^β and the concrete exponentiation $\mathsf{exp}\,(\alpha, \beta)$ are well defined and well behaved in this case.

Theorem 24 (⚙). *For all ordinals α and β such that α has a trichotomous least element, we have*

$$\alpha^\beta = \mathsf{exp}\,(\alpha, \beta).$$

Proof. Let $\bot$ be the trichotomous least element of α. We prove the equation by transfinite induction on β. Our induction hypothesis reads:

$$\forall (b : \beta).\, \alpha^{\beta \downarrow b} = \mathsf{exp}\,(\alpha, \beta \downarrow b)\,. \qquad \text{(IH)}$$

These equalities induce simulations and simulations preserve initial segments (Lemma 1), so for all $b : \beta$, the simulation provides for every $e : \alpha^{\beta \downarrow b}$ a unique $l : \mathsf{exp}\,(\alpha, \beta \downarrow b)$ with $\alpha^{\beta \downarrow b} \downarrow e = \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow l$, and similarly if we start with an element of $\mathsf{exp}\,(\alpha, \beta \downarrow b)$ instead.

By extensionality of the ordinal of ordinals, it suffices to show that each initial segment of α^β is equal to an initial segment of $\mathsf{exp}\,(\alpha, \beta)$ and vice versa.

Suppose first that we have $e_0 : \alpha^\beta$. By Eq. (3) we have $e_0 = [\mathsf{inl} \star, \star] \equiv \bot$ or $e_0 = [\mathsf{inr}\, b, (e, a)]$ with $a : \alpha$ and $e : \alpha^{\beta \downarrow b}$. In the first case, we have $\alpha^\beta \downarrow e_0 = \mathbf{0} = \mathsf{exp}\,(\alpha, \beta) \downarrow []$. The second case has two subcases: a is either equal to the trichotomous least element $\bot$, or greater than it. If $a = \bot$, then we calculate

$$\begin{aligned} &\alpha^\beta \downarrow e_0 \\ &= \alpha^{\beta \downarrow b} \downarrow e && \text{(by Proposition 7)} \\ &= \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow l && \text{(for a unique } l \text{ by IH)} \\ &= \mathsf{exp}\,(\alpha, \beta) \downarrow \iota_b\, l && \text{(using Lemma 1 and } \iota_b \text{ from Eq. (4))} \end{aligned}$$

completing the proof for this case. If $a > \bot$, then we calculate

$$\begin{aligned} &\alpha^\beta \downarrow e_0 \\ &= \alpha^{\beta \downarrow b} \times (\alpha \downarrow a) + \alpha^{\beta \downarrow b} \downarrow e && \text{(by Proposition 7)} \\ &= \mathsf{exp}\,(\alpha, \beta \downarrow b) \times (\alpha \downarrow a) \\ &\quad + \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow l && \text{(for a unique } l \text{ by IH)} \\ &= \mathsf{exp}\,(\alpha, \beta) \downarrow ((a, b) :: \iota_b\, l) && \text{(by Proposition 19)} \end{aligned}$$

completing the proof for this case.

Now let $l_0 : \mathsf{exp}\,(\alpha, \beta)$. Then either $l_0 = []$ in which case we are done, because $\mathsf{exp}\,(\alpha, \beta) \downarrow l_0 = \mathbf{0} = \alpha^\beta \downarrow \bot$, or $l_0 = (a, b) :: l$. In this second case, we calculate

$$\begin{aligned} &\mathsf{exp}\,(\alpha, \beta) \downarrow ((a, b) :: l) \\ &= \mathsf{exp}\,(\alpha, \beta \downarrow b) \times (\alpha \downarrow a) \\ &\quad + \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow \tau_b\, l && \text{(by Proposition 19)} \\ &= \alpha^{\beta \downarrow b} \times (\alpha \downarrow a) + \alpha^{\beta \downarrow b} \downarrow e && \text{(for a unique } e \text{ by IH)} \\ &= \alpha^\beta \downarrow [\mathsf{inr}\, b, (e, a)] && \text{(by Proposition 7)} \end{aligned}$$

finishing the proof. □

The following decidability properties follow directly from Theorem 24 and Propositions 22 and 23.

Corollary 25 (⚙).

- *Suppose that α has a trichotomous least element. If α and β have decidable equality, then so does α^β.*
- *Suppose α has a least element. If α and β are trichotomous, then so is α^β.* □

Before we knew that Theorem 24 was true, we started working on a direct proof that repeated concrete exponentiation is exponentiation by the product, i.e., $\mathsf{exp}\,(\mathsf{exp}\,(\alpha, \beta), \gamma) = \mathsf{exp}\,(\alpha, \beta \times \gamma)$. However, dealing with all of the side conditions stating that lists are decreasing proved to be too tedious for us to finish the construction. Fortunately, this result follows for free via Proposition 11 and Theorem 24.

Corollary 26 (). *For ordinals α, β and γ with α having a trichotomous least element, we have* $\mathsf{exp}\,(\alpha, \beta \times \gamma) = \mathsf{exp}\,(\mathsf{exp}\,(\alpha, \beta), \gamma)$. □

Note that in the above corollary we implicitly used that $\mathsf{exp}(\alpha, \beta)$ has a trichotomous least element (Proposition 17).

B. Lists as Representations

As remarked above, we used the assumption that the least element of α is trichotomous to show that the set of lists $[\alpha, \beta]_<$ is an ordinal. But even without this assumption, we can view such a list as a *representation* of something in the abstract exponential α^β. This is made precise by the following denotation function.

Definition 27 (Denotation function, $[\![-]\!]_\beta$). We define

$$[\![-]\!]_\beta : [\alpha, \beta]_< \to \alpha^\beta$$

for any α by transfinite induction on β:

$$\begin{aligned} [\![\, [\,] \,]\!]_\beta &:\equiv \bot; \\ [\![(a, b) :: l]\!]_\beta &:\equiv [\mathsf{inr}\, b, ([\![\tau_b\, l]\!]_{\beta\downarrow b}, a)]; \end{aligned}$$

with τ_b as in Eq. (5).

Remark 28. Note that the above definition is not by recursion on the list, as the recursive call in the non-empty list case is on $\tau_b\, l$, which is not directly structurally smaller than $(a, b) :: l$. With more work, $[\![l]\!]_\beta$ could be defined by induction on the *length* of the list l, but a construction by transfinite induction on β with a non-recursive case-split on l is more straightforward.

Every element of the abstract exponential α^β merely has a representation as a list, in the following sense:

Proposition 29 (). *For all ordinals α and β, the denotation function $[\![-]\!]_\beta$ is surjective.*

Proof. We do transfinite induction on β. For every $x : \alpha^\beta$ we need to show that there exists a list l with $[\![l]\!]_\beta = x$. As the goal is a proposition, we can do case distinction on x; for $x = [\mathsf{inl}\,\star, \star]$, the list is $[\,]$, which leaves us with the case that x is given by $b : \beta$ together with $e : \alpha^{\beta\downarrow b}$ and $a : \alpha$. By the induction hypothesis, there exists a list $l' : [\alpha, \beta \downarrow b]_<$ whose denotation is e, ensuring that $(a, b) :: \iota_b\, l'$ represents x. □

Note that Proposition 29 does not assume that α has a least element, and definitely not that it has a trichotomous least element. However, when α does have a trichotomous least element, the equality established in Theorem 24 induces a map from concrete to abstract exponentials,

$$\mathsf{con\text{-}to\text{-}abs} : \mathsf{exp}\,(\alpha, \beta) \to \alpha^\beta,$$

and we could hope to relate this function to the denotation function $[\![-]\!]_\beta : [\alpha, \beta]_< \to \alpha^\beta$.

Thanks to the trichotomous least element, we can normalize a list by removing those pairs which have the least element in the α-component, yielding a map

$$\mathsf{normalize} : [\alpha, \beta]_< \to [\alpha_{>\bot}, \beta]_<$$

with

$$\mathsf{normalize}\,((a, b) :: l) :\equiv \begin{cases} \mathsf{normalize}\, l & \text{if } a = \bot; \\ (a, b) :: \mathsf{normalize}\, l & \text{if } a > \bot. \end{cases}$$

Note that the codomain of normalize is exactly $\mathsf{exp}\,(\alpha, \beta)$.

The normalization function allows us to compare the induced map with the denotation function:

Theorem 30 (). *In case α has a trichotomous least element, the denotation function coincides with the equality between abstract and concrete exponentiation in the following sense:*

$$[\![-]\!]_\beta = \mathsf{con\text{-}to\text{-}abs} \circ \mathsf{normalize}.$$

The two following lemmas directly prove the theorem.

Lemma 31 (). *The induced map* con-to-abs *coincides with a denotation function*

$$[\![-]\!]'_\beta : [\alpha_{>\bot}, \beta]_< \to \alpha^\beta$$

that is defined by transfinite induction just like $[\![-]\!]_\beta$ but with a restricted domain instead.

Proof. We prove $\mathsf{con\text{-}to\text{-}abs}\, l = [\![l]\!]'_\beta$ for all $l : [\alpha_{>\bot}, \beta]_<$ by induction on β and a case distinction on l, and use that elements of ordinals are equal if and only if they determine the same initial segments. The case of the empty list is easy, as it is the least element of $[\alpha_{>\bot}, \beta]_<$, and since con-to-abs is a simulation it must map it to the least element of α^β. For nonempty lists, we have

$$\begin{aligned} &\alpha^\beta \downarrow \mathsf{con\text{-}to\text{-}abs}\,((a, b) :: l) \\ &= \mathsf{exp}\,(\alpha, \beta) \downarrow ((a, b) :: l) && \text{(I)} \\ &= \mathsf{exp}\,(\alpha, \beta \downarrow b) \times (\alpha \downarrow a) + \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow \tau_b\, l && \text{(II)} \\ &= \alpha^{\beta\downarrow b} \times (\alpha \downarrow a) + \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow \tau_b\, l && \text{(III)} \\ &= \alpha^{\beta\downarrow b} \times (\alpha \downarrow a) + \alpha^{\beta\downarrow b} \downarrow \mathsf{con\text{-}to\text{-}abs}\,(\tau_b\, l) && \text{(IV)} \\ &= \alpha^{\beta\downarrow b} \times (\alpha \downarrow a) + \alpha^{\beta\downarrow b} \downarrow [\![\tau_b\, l]\!]'_{\beta\downarrow b} && \text{(V)} \\ &= \alpha^\beta \downarrow [\mathsf{inr}\, b, ([\![\tau_b\, l]\!]'_{\beta\downarrow b}, a)] && \text{(VI)} \\ &\equiv \alpha^\beta \downarrow [\![(a, b) :: l]\!]'_\beta, \end{aligned}$$

where steps (I) and (IV) use that simulations preserve initial segments, step (II) is by Proposition 19, step (III) is by Theorem 24, step (V) uses the induction hypothesis, and (VI) is by Proposition 7. □

Lemma 32 (). *The denotations are related via normalization:*

$$[\![-]\!]_\beta = [\![-]\!]'_\beta \circ \mathsf{normalize}.$$

Proof. This is proved by transfinite induction on β. The case of the empty list is easy. So consider an element of $[\alpha, \beta]_<$ of the form $(a, b) :: l$. Suppose first that a is not the trichotomous least element $\bot$, then

$$\begin{aligned} [\![(a, b) :: l]\!]_\beta &\equiv [\mathsf{inr}\, b, ([\![\tau_b\, l]\!]_{\beta\downarrow b}, a)] \\ &= [\mathsf{inr}\, b, ([\![\mathsf{normalize}(\tau_b\, l)]\!]'_{\beta\downarrow b}, a)] && \text{(by IH)} \\ &= [\mathsf{inr}\, b, ([\![\tau_b\, (\mathsf{normalize}\, l)]\!]'_{\beta\downarrow b}, a)] \\ &\equiv [\![\mathsf{normalize}((a, b) :: l)]\!]'_\beta, \end{aligned}$$

where the penultimate equality uses that $\mathsf{normalize}$ and τ_b commute.

For the trichotomous least element $\bot$, we have

$$\begin{aligned}
[\![(\bot, b) :: l]\!]_\beta &\equiv [\mathsf{inr}\, b, ([\![\tau_b\, l]\!]_{\beta\downarrow b}, \bot)] \\
&= [\![\iota_b(\mathsf{normalize}(\tau_b\, l))]\!]'_\beta && \text{(I)} \\
&= [\![\mathsf{normalize}(\iota_b(\tau_b\, l))]\!]'_\beta && \text{(II)} \\
&= [\![\mathsf{normalize}\, l]\!]'_\beta && \text{(III)} \\
&\equiv [\![\mathsf{normalize}((\bot, b) :: l)]\!]'_\beta,
\end{aligned}$$

where (II) holds because $\mathsf{normalize}$ and ι_b commute, (III) because τ_b cancels ι_b, and (I) holds because these elements determine the same initial segments:

$$\begin{aligned}
&\alpha^\beta \downarrow [\mathsf{inr}\, b, ([\![\tau_b\, l]\!]_{\beta\downarrow b}, \bot)] \\
&= \alpha^{\beta\downarrow b} \times (\alpha \downarrow \bot) + \alpha^{\beta\downarrow b} \downarrow [\![\tau_b\, l]\!]_{\beta\downarrow b} && \text{(by Proposition 7)} \\
&= \alpha^{\beta\downarrow b} \downarrow [\![\tau_b\, l]\!]_{\beta\downarrow b} && (\text{as } \alpha \downarrow \bot = \mathbf{0}) \\
&= \alpha^{\beta\downarrow b} \downarrow [\![\mathsf{normalize}(\tau_b\, l)]\!]'_{\beta\downarrow b} && \text{(by IH)} \\
&= \mathsf{exp}\,(\alpha, \beta \downarrow b) \downarrow \mathsf{normalize}(\tau_b\, l) && \text{(by (i) in Lemma 1)} \\
&= \alpha^{\beta\downarrow b} \downarrow [\![\iota_b(\mathsf{normalize}(\tau_b\, l))]\!]'_\beta, && \text{(by (i) in Lemma 1)}
\end{aligned}$$

where the second-to-last equality employs the map $[\![-]\!]'_{\beta\downarrow b}$, which is a simulation by Lemma 31 as con-to-abs is induced by an equality (and hence is a simulation). For the final equality we consider the composition of simulations $[\![-]\!]'_\beta \circ \iota_b$. □

Remark 33. We do not know of a direct proof that the denotation function con-to-abs is an ordinal equivalence, or even a simulation. Instead, our proof of Theorem 24 makes use of the equalities given by the induction hypothesis to prove the inductive step.

VI. On Grayson's Decreasing Lists

A slight variation of our decreasing list construction $\mathsf{exp}\,(\alpha, \beta)$ was suggested in Grayson's PhD thesis [18, §IX.3], the relevant part of which has appeared as Grayson [19, §3.2]. His setting is an unspecified version of constructive set (or type) theory and uses setoids (i.e., sets with an equivalence relation) for the construction. Grayson's suggestion does not require the base ordinal α to have a least element. Unfortunately, his construction (which is presented without proofs) does not work in the claimed generality, as assuming that it always yields an ordinal is equivalent to the law of excluded middle. We present our argument for this in the setting of the current paper, but the argument carries over to a foundation based on setoids.

Definition 34 (Grayson [18, §IX.3] ⚙)**.** Given an ordinal α, we say that $x : \alpha$ is *positively non-minimal* if $\exists(a : \alpha).x > a$. Then, given a second ordinal β, the type of *Grayson lists*, written $\mathsf{Gr}(\alpha, \beta)$, is the type of lists over $\alpha \times \beta$ that are decreasing in the β-component and where all α-components are positively non-minimal.

If α has a trichotomous least element $\bot$, it is easy to see (and formalized [29, Grayson]) that x is positively non-minimal if and only if $x > \bot$, i.e. the two notions of positivity coincide. Thus, under this assumption, $\mathsf{Gr}(\alpha, \beta)$ becomes equivalent to $\mathsf{exp}\,(\alpha, \beta)$. However, the assumption cannot be removed:

Proposition 35 (⚙)**.** LEM *holds if and only if* $\mathsf{Gr}(\alpha, \beta)$ *is an ordinal for all (possibly large) ordinals* α *and* β*. This remains true with the additional condition that* α *has a least element.*

Proof. If LEM is assumed, then every α is empty or has a trichotomous least element, and the construction works in either case. For the other direction, let us fix β to be $\mathbf{1}$; then, $\mathsf{Gr}(\alpha, \mathbf{1})$ is equivalent to $\mathbf{1} + \alpha^+$, where the latter is the type of positively non-minimal elements of α. It is easy to see that $\mathsf{Gr}(\alpha, \mathbf{1})$ is an ordinal if and only if α^+ is. But the assumption that Ord^+ is an ordinal implies LEM (cf. the proof of Proposition 14). □

Finally, Grayson [18, §IX.3] claims the recursive equation

$$\mathsf{Gr}(\alpha, \beta) = \mathbf{1} \vee \mathsf{sup}_{b:\beta}(\mathsf{Gr}(\alpha, \beta \downarrow b) \times \alpha)$$

in full generality. It follows from the above discussion and Theorem 24 that this equation holds if α has a detachable least element. The latter condition is indispensable, as the left-hand side always has a least detachable element (the empty list), while, for $\beta = \mathbf{1}$ and $\alpha \geq \mathbf{1}$, the right-hand side does so exactly if α does. For more details, we refer the interested reader to the `Grayson` file of our formalization [29].

VII. Constructive Taboos

We were able to give constructive proofs of many desirable properties of ordinal exponentiation for both abstract and concrete exponentials, e.g., monotonicity in the exponent, or algebraic laws such as $\alpha^{\beta+\gamma} = \alpha^\beta \times \alpha^\gamma$ and $\alpha^{\beta\times\gamma} = (\alpha^\beta)^\gamma$. In this section, we explore classical properties that are not possible to prove constructively. A first example is monotonicity in the base, which is inherently classical.

Proposition 36 (⚙)**.** *Exponentiation is monotone in the base if and only if* LEM *holds. In fact,* LEM *is already implied by each of the following weaker statements, even when* α *and* β *are each assumed to have a trichotomous least element:*

(i) $\alpha < \beta \to \alpha^\gamma \leq \beta^\gamma$;
(ii) $\alpha < \beta \to \alpha \times \alpha \leq \beta \times \beta$.

Proof. Note that (i) implies (ii) by taking $\gamma :\equiv \mathbf{2}$. To see that (ii) implies LEM, we consider an arbitrary proposition P and the ordinals $\alpha :\equiv \mathbf{2}$ and $\beta :\equiv \mathbf{3} + P$. Clearly, $\alpha < \beta$ and α and β respectively have trichotomous least elements 0 and $\mathsf{inl}\, 0$, so that we get a simulation $f : \alpha \times \alpha \to \beta \times \beta$ by assumption.

If we have $p : P$, then $g_p : \alpha \times \alpha \to \beta \times \beta$ defined by

$$\begin{aligned}
g_p(0,0) &:\equiv (\mathsf{inl}\, 0, \mathsf{inl}\, 0), & g_p(1,0) &:\equiv (\mathsf{inl}\, 1, \mathsf{inl}\, 0), \\
g_p(0,1) &:\equiv (\mathsf{inl}\, 2, \mathsf{inl}\, 0), & g_p(1,1) &:\equiv (\mathsf{inr}\, p, \mathsf{inl}\, 0).
\end{aligned}$$

can be checked to be a simulation.

Since simulations are unique, g_p must agree with f in case P holds. We now simply check where $(1, 1)$ gets mapped by f: if it is of the form $(\mathsf{inr}\, p, y')$ then obviously P holds; and if it is of the form $(\mathsf{inl}\, y, y')$, then we claim that $\neg P$ holds. Indeed, assuming $p : P$ we obtain $(\mathsf{inl}\, y, y') = f(1, 1) = g_p(1, 1) \equiv (\mathsf{inr}\, p, \mathsf{inl}\, 0)$, which is impossible as coproducts are disjoint. □

We note that the argument above works for any operation satisfying the exponentiation specification (by Lemma 3).

The following is an example of an equation that does not follow from the specification of exponentiation, since we cannot decide if a given proposition is zero, a successor, or a nontrivial supremum. Nevertheless, it is true and it is used to derive a taboo below.

Lemma 37 (⚙). *For a proposition P we have $\mathbf{2}^P = \mathbf{1} + P$.*

Proof. Given $p : P$, we note that $\mathbf{2}^{P \downarrow p} \times \mathbf{2} = \mathbf{2}^{\mathbf{0}} \times \mathbf{2} = \mathbf{2}$, so that $\mathbf{2}^P = \sup F_\bullet$ with $F : \mathbf{1} + P \to \mathsf{Ord}$ defined as $F(\mathsf{inl}\, \star) :\equiv \mathbf{1}$ and $F(\mathsf{inr}\, p) :\equiv \mathbf{2}$. Since sup gives the least upper bound and P implies $\mathbf{1} + P = \mathbf{2}$, we get a simulation $\sup F_\bullet \le \mathbf{1} + P$. Conversely, we note that

$$(\mathbf{1} + P) \downarrow (\mathsf{inl}\, \star) = \mathbf{0} = \mathbf{1} \downarrow \star = \sup F_\bullet \downarrow [\mathsf{inl}\, \star, \star], \text{ and}$$
$$(\mathbf{1} + P) \downarrow (\mathsf{inr}\, p) = \mathbf{1} = \mathbf{2} \downarrow 1 = \sup F_\bullet \downarrow [\mathsf{inr}\, p, 1],$$

yielding a simulation $\mathbf{1} + P \le \sup F_\bullet$. □

As mentioned just after Lemma 1, showing that $\alpha \le \beta$ is often straightforward in a classical metatheory, as LEM ensures that a simulation can always be "carved out" out of any order preserving function $\alpha \to \beta$. This result is unavoidably classical, in the sense that it in turn implies the law of excluded middle.

Lemma 38 (⚙). *Every order preserving function between ordinals induces a simulation if and only if* LEM *holds.*

Proof. The right-to-left direction was proven and formalized by Escardó [10, Ordinals.OrdinalOfOrdinals]: assuming LEM, to prove $\alpha \le \beta$ it suffices to show that it is impossible that $\beta < \alpha$, but this readily follows from having an order preserving function from α to β.

In the other direction, let P be an arbitrary proposition, and consider $\alpha :\equiv \mathbf{1}$ and $\beta :\equiv P + \mathbf{1}$. The function $\star \mapsto \mathsf{inr}\, \star : \alpha \to \beta$ is trivially order preserving, and thus gives rise to a simulation $f : \alpha \to \beta$ by assumption. Since simulations preserve least elements, we can then decide P, since P holds if and only of $f\, \star = \mathsf{inl}\; p$ for some $p : P$. □

The following proposition is another example of a useful property of exponentials which is classically true, but unfortunately impossible to realize constructively.

Proposition 39 (⚙). *The following are equivalent:*

(i) for all ordinals β, we have $\beta \le \mathbf{2}^\beta$;
(ii) for all ordinals β and $\alpha > \mathbf{1}$, we have $\beta \le \alpha^\beta$;
(iii) LEM.

Proof. Clearly, (ii) implies (i). To see that (i) implies (iii), we let P be an arbitrary proposition and we consider $\alpha :\equiv \mathbf{2}$ and $\beta :\equiv P + \mathbf{1}$. By Lemma 37 we have $\alpha^\beta = (1 + P) \times \mathbf{2}$, so that we get a simulation $f : P + \mathbf{1} \to (1 + P) \times \mathbf{2}$ by assumption. We now decide P by inspecting $f(\mathsf{inr}\, \star)$.

If $f(\mathsf{inr}\, \star) = (\mathsf{inr}\, p, b)$, then obviously P holds.

If $f(\mathsf{inr}\, \star) = (\mathsf{inl}\, \star, 0)$, then $\neg P$ holds, for if we had $p : P$, then, since simulations preserve the least element, $f(\mathsf{inl}\, p) = (\mathsf{inl}\, \star, 0) = f(\mathsf{inr}\, \star)$ which is impossible as f is injective (by Lemma 1) and coproducts are disjoint.

Finally, if $f(\mathsf{inr}\, \star) = (\mathsf{inl}\, \star, 1)$, then P holds, because f is a simulation and $(\mathsf{inl}\, \star, 0) < (\mathsf{inl}\, \star, 1) = f(\mathsf{inr}\, \star)$, so there must be $x : P + \mathbf{1}$ with $x < \mathsf{inr}\, \star$ and $f\, x = (\mathsf{inl}\, \star, 0)$. But then x must be of the form $\mathsf{inl}\, p$.

Finally, to prove (iii) implies (ii), assume $\alpha > \mathbf{1}$, i.e., there is $a_1 : \alpha$ such that $\alpha \downarrow a_1 = \mathbf{1}$, and assume LEM. By Lemma 38, $\beta \le \alpha^\beta$ holds as soon as we have an order preserving map $f : \beta \to \alpha^\beta$. We claim that the map $f\, b :\equiv [\mathsf{inr}\, b, (\bot, a_1)]$ does the job. Indeed, by Proposition 7, we have $\alpha^\beta \downarrow f\, b = \alpha^{\beta \downarrow b}$, so that if $b < b'$, then $\alpha^{\beta \downarrow b} < \alpha^{\beta \downarrow b'}$ by Proposition 8 and hence $f\, b < f\, b'$ as taking initial segments is order reflecting. □

VIII. Conclusions and Future Work

Working in homotopy type theory, we have presented two constructively well behaved definitions of ordinal exponentiation and showed them to be equivalent in case the base ordinal has a trichotomous least element. The equivalence, in combination with the univalence axiom, was used to transfer various results, such as algebraic laws and decidability properties, from one construction to the other. We furthermore marked the limits of a constructive treatment by presenting no-go theorems that show the law of excluded middle (LEM) to be equivalent to certain statements about ordinal exponentiation.

A natural question, to which we do not yet have a conclusive answer, is whether it is possible to fuse the two constructions of this paper and define ordinal exponentiation for base ordinals that do not necessarily have a trichotomous least element via quotiented lists.

In future work, we would like to develop constructive ordinal arithmetic further by studying subtraction, division and logarithms. Like with exponentiation, a careful treatment is required here, as e.g., the existence of ordinal subtraction in the usual formulation is equivalent to excluded middle, as observed by Escardó [10, Ordinals.AdditionProperties]. It is possible, however, to construct a function $f : \mathsf{Ord} \times \mathsf{Ord} \to \mathsf{Ord}$ satisfying the weaker requirement that $f(\alpha, \beta)$ is the greatest ordinal γ with $\alpha + \gamma \le \beta$ and $\gamma \le \beta$. In fact, this is actually an instance of a more general construction: given a property P of Ord that is continuous, antitone and bounded, then there is a maximal ordinal satisfying P. Here a property is *antitone* if $\alpha \le \beta$ gives $P\, \beta \to P\, \alpha$ and *bounded* if there exists δ such that $\alpha \le \delta$ whenever α satisfies P. This more general construction can be applied to division and logarithms, and is a constructive reformulation of a well known theorem in classical ordinal theory, where the boundedness condition is redundant, see e.g., Enderton [12, Theorem Schema 8D].

Acknowledgements and Funding

We are grateful to Martín Escardó, Paul Levy and David Wärn for helpful discussions on ordinals. In particular, Escardó pointed us to the work of Grayson, while the definition of abstract exponentiation builds on a suggestion by Wärn. We appreciate the pointers and suggestions by the anonymous reviewers that helped us to improve the paper. We also thank the participants, organizers and support staff of the HIM trimester programme *Prospects of Formal Mathematics* for the opportunity to work on this project during our stay.

This work was supported by The Royal Society (grant references URF\R1\191055, RF\ERE\210032, RF\ERE\231052, URF\R\241007), the UK National Physical Laboratory Measurement Fellowship project "Dependent types for trustworthy tools", and the Engineering and Physical Sciences Research Council (EP/Y000455/1). The HIM programme was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2047/1 – 390685813.

All data underpinning this publication are openly available from Zenodo at doi:10.5281/zenodo.15461750 [29].

References

[1] M. Rathjen, "The art of ordinal analysis," in *Proceedings of the International Congress of Mathematicians*. European Mathematical Society Publishing House, 2007, vol. 2, pp. 45–69.

[2] R. W. Floyd, "Assigning meanings to programs," in *Mathematical Aspects of Computer Science*, ser. Proceedings of Symposia in Applied Mathematics, J. T. Schwartz, Ed., vol. 19. American Mathematical Society, 1967, pp. 19–32.

[3] P. Aczel, "An introduction to inductive definitions," in *Handbook of Mathematical Logic*, ser. Studies in Logic and the Foundations of Mathematics, J. Barwise, Ed. North-Holland Publishing Company, 1977, vol. 90, pp. 739–782.

[4] P. Dybjer and A. Setzer, "A finite axiomatization of inductive-recursive definitions," in *Typed Lambda Calculi and Applications*, ser. Lecture Notes in Computer Science, J.-Y. Girard, Ed., vol. 1581. Springer, 1999, pp. 129–146.

[5] F. Nordvall Forsberg, C. Xu, and N. Ghani, "Three equivalent ordinal notation systems in cubical Agda," in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, J. Blanchette and C. Hriţcu, Eds. Association for Computing Machinery, 2020, pp. 172–185.

[6] J. Eremondi, "Strictly monotone Brouwer trees for well founded recursion over multiple arguments," in *Proceedings of the 13th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2024)*. Association for Computing Machinery, 2024, pp. 205–217.

[7] P. Martin-Löf, *Notes on constructive mathematics*. Almqvist & Wiksell, 1970.

[8] T. Coquand, H. Lombardi, and S. Neuwirth, "Constructive theory of ordinals," in *Mathematics for Computation*, M. Benini, O. Beyersdorff, M. Rathjen, and P. Schuster, Eds. World Scientific, 2023, pp. 287–318.

[9] Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[10] M. H. Escardó *et al.*, "Ordinals in univalent type theory in Agda notation," Since 2018, Agda development, HTML rendering available at: https://www.cs.bham.ac.uk/~mhe/TypeTopology/Ordinals.index.html. [Online]. Available: https://github.com/martinescardo/TypeTopology

[11] P. Halmos, *Naive Set Theory*. Springer, 1960.

[12] H. B. Enderton, *Elements Of Set Theory*. Academic Press, 1977.

[13] J. G. Rosenstein, *Linear Orderings*. Academic Press, 1982.

[14] M. Holz, K. Steffens, and E. Weitz, *Introduction to Cardinal Arithmetic*. Birkhäuser, 1999.

[15] T. Jech, *Set theory*. Springer, 2003.

[16] W. Pohlers, *Proof Theory: the First Step into Impredicativity*. Springer, 2009.

[17] W. Sierpiński, *Cardinal and Ordinal Numbers*, ser. Monografie Matematyczne. Państwowe Wydawnictwo Naukowe, 1958, vol. 34.

[18] R. J. Grayson, "Intuitionistic set theory," Ph.D. dissertation, University of Oxford, 1978.

[19] ——, "Constructive well-orderings," *Mathematical Logic Quarterly*, vol. 28, no. 33–38, pp. 495–504, 1982.

[20] G. Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre. II," *Mathematische Annalen*, vol. 49, no. 2, pp. 207–246, 1897.

[21] N. Kraus, F. Nordvall Forsberg, and C. Xu, "Type-theoretic approaches to ordinals," *Theoretical Computer Science*, vol. 957, 2023.

[22] A. S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, 2nd ed. Cambridge University Press, 2000.

[23] A. Church, "The constructive second number class," *Bulletin of the American Mathematical Society*, vol. 44, pp. 224–232, 1938.

[24] S. C. Kleene, "On notation for ordinal numbers," *The Journal of Symbolic Logic*, vol. 3, no. 4, pp. 150–155, 1938.

[25] A. Setzer, "Proof theory of Martin-Löf Type Theory. An overview," *Mathématiques et sciences humaines*, vol. 165, pp. 59–99, 2004.

[26] P. Hancock, "(Ordinal-theoretic) proof theory," Course notes from Midlands Graduate School. Available at https://martinescardo.github.io/events/MGS08/notes/proofTheory.pdf, 2008.

[27] The mathlib Community, "The Lean mathematical library," in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*. Association for Computing Machinery, 2020, pp. 367–381.

[28] J. C. Blanchette, A. Popescu, and D. Traytel, "Cardinals in Isabelle/HOL," in *Interactive Theorem Proving*, G. Klein and R. Gamboa, Eds. Springer, 2014, pp. 111–127.

[29] T. de Jong, N. Kraus, F. Nordvall Forsberg, and C. Xu, "Agda formalization of *Ordinal Exponentiation in Homotopy Type Theory*," https://cs.bham.ac.uk/~mhe/TypeTopology/Ordinals.Exponentiation.index.html, 2025, archived as doi:10.5281/zenodo.15461750. [Online]. Available: https://github.com/martinescardo/TypeTopology/

[30] ——, "Set-theoretic and type-theoretic ordinals coincide," in *38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2023)*, 2023, pp. 1–13.

[31] T. de Jong and M. H. Escardó, "On small types in univalent foundations," *Logical Methods in Computer Science*, vol. 19, no. 2, pp. 8:1–8:33, 2023.

[32] T. de Jong and M. H. Escardó, "Domain theory in univalent foundations II: Continuous and algebraic domains," 2024, arXiv[cs.LO]: 2407.06956.

[33] T. Coquand and A. Spiwack, "Constructively finite?" in *Contribuciones científicas en honor de Mirian Andrés Gómez*. Universidad de La Rioja, 2010, pp. 217–130.

[34] D. Frumin, H. Geuvers, L. Gondelman, and N. van der Weide, "Finite sets in homotopy type theory," in *Proceedings of the 7th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2018)*. Association for Computing Machinery, 2018, pp. 201–214.