# Interpreting inductive-inductive definitions as indexed inductive definitions

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(Work in progress)

# What is induction-induction?

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- Induction-induction is an induction principle in Martin-Löf Type Theory.
- It allows us to define $A : \mathrm{Set}$, together with $B : A \to \mathrm{Set}$, where:
  - Both $A$ and $B(a)$ for $a : A$ are inductively defined.
  - The constructors for $A$ can refer to $B$ and vice versa.
  - The constructors for $B$ can also use constructors for $A$.

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An inductive-inductive definition is in general not:

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- An inductive-recursive definition
  - Because $B : A \to \mathrm{Set}$ is defined inductively, not recursively.

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- An ordinary mutual inductive definition
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- An inductive-recursive definition
  - Because $B : A \to \mathrm{Set}$ is defined inductively, not recursively.
- An indexed inductive definition
  - Because the index set $A : \mathrm{Set}$ is defined along with $B : A \to \mathrm{Set}$, and not fixed beforehand.
  - However, we will show that it can be reduced to IID.

# Habitat 67

2600 Avenue Pierre Dupoy, Montréal, Quebec, Canada

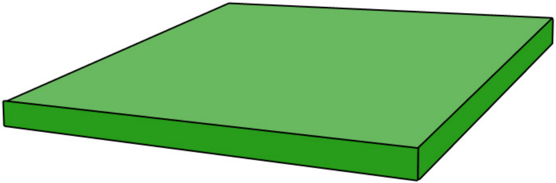
# Modelling Habitat 67

$$\mathsf{Platform} : \mathsf{Set} \qquad \mathsf{Building} : \mathsf{Platform} \to \mathsf{Set}$$

- $p$ : Platform means $p$ is a platform.
- $b$ : Building($p$) means $b$ is a building built on the platform $p$.

# Example: buildings and platforms

$$\frac{}{\mathsf{ground} : \mathsf{Platform}}$$



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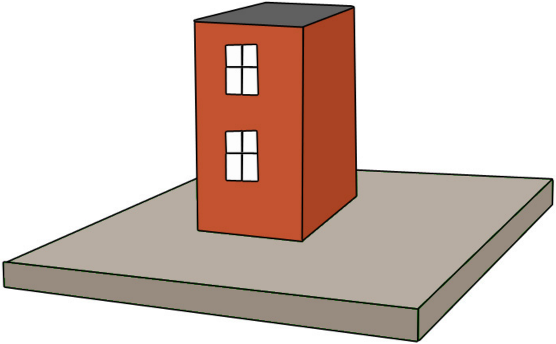
$$\frac{p : \mathrm{Platform}}{\mathrm{onTop}(p) : \mathrm{Building}(p)}$$



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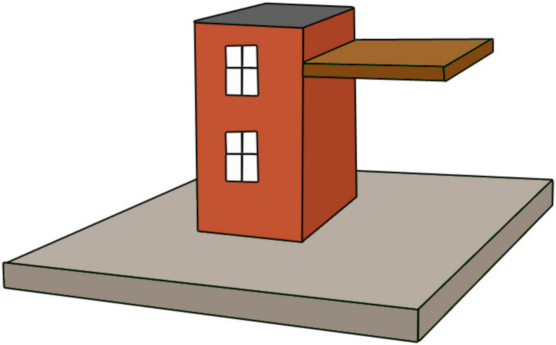

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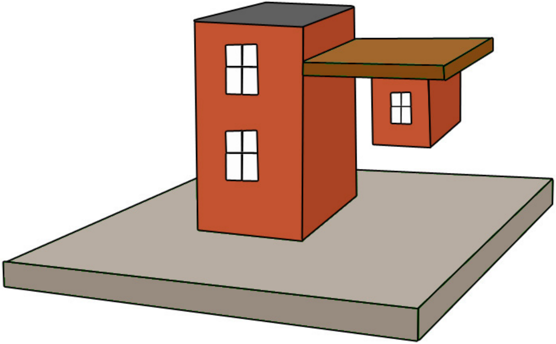
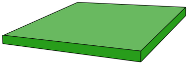
$$\frac{p : \mathsf{Platform} \qquad b : \mathsf{Building}(p)}{\mathsf{hangingUnder}(p, b) : \mathsf{Building}(\mathsf{extension}(p, b))}$$



# Buildings and platforms



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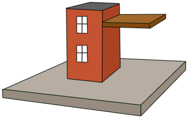
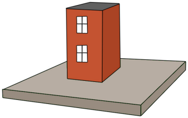


$$\frac{p : \mathrm{Platform} \qquad b : \mathrm{Building}(p)}{\mathrm{extension}(p, b) : \mathrm{Platform}}$$



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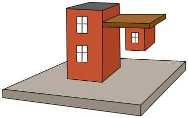
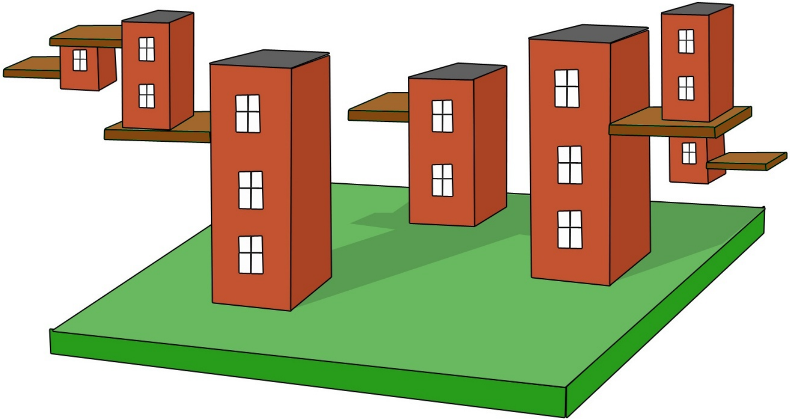


$$\frac{p : \mathrm{Platform} \qquad b : \mathrm{Building}(p)}{\mathrm{hangingUnder}(p, b) : \mathrm{Building}(\mathrm{extension}(p, b))}$$

# . . . and so on



# More seriously

On a more serious note, instances of induction-induction have been used implicitly by

- Dybjer (1996),
- Danielsson (2007), and
- Chapman (2009)

to model dependent type theory inside itself.

# Type theory inside type theory

- Context : Set
- Type : Context $\to$ Set
- Term : $(\Gamma$ : Context$) \to$ Type$(\Gamma) \to$ Set
- ...
- Substitutions, ...
- ...

defined inductively

# An axiomatisation

We have given an axiomatisation of inductive-inductive definitions.

- Similar to axiomatisation of induction-recursion by Dybjer and Setzer.
- Main idea: add universe $U$ consisting of codes for ind.-ind. defined sets.
  - The codes reflect syntactic definition of the sets.
  - For each $\gamma : U$, there is $A_\gamma : \text{Set}$, $B_\gamma : A_\gamma \to \text{Set}$.
  - Appropriate introduction and elimination rules (stating $A_\gamma$, $B_\gamma$ inductively defined).

Makes meta-mathematical analysis of the theory of *all* inductive-inductive definitions possible.

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- Yes, we have a set-theoretical model.

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## What about the proof theoretical strength?

- We will show that induction-induction can be reduced to indexed inductive definitions.
- Hence these theories have the same proof theoretical strength.

## But is it consistent?

- Yes, we have a set-theoretical model.
- More satisfying answer: yes, we have a model in $\mathrm{IID}^{\mathrm{ext}}$.

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- We will show that induction-induction can be reduced to indexed inductive definitions.
- Hence these theories have the same proof theoretical strength.

# Reduction to indexed inductive definitions

# What are indexed inductive definitions?

- An inductive family of sets (for fixed index set $I$).
- Typical examples:
  - finite sets $\mathsf{Fin} : \mathbb{N} \to \mathsf{Set}$.
  - vectors (lists of certain length) $\mathsf{Vec} : \mathbb{N} \to \mathsf{Set}$.
- Whole family defined at once, so constructors can relate different indices.
- Special case: mutual definitions – indexed by finite set.
- Get axiomatisation "for free" by considering Dybjer and Setzer's axiomatisation of indexed IR with trivial recursive part ($T : U \to \mathbf{1}$).

# The general picture

- Have:

$$A : \text{Set} \qquad B : A \to \text{Set}$$

- Will define (with IID):

$$A_{\text{pre}} : \text{Set} \qquad B_{\text{pre}} : \text{Set}$$

as a first approximation, and then

$$\text{good}A : A_{\text{pre}} \to \text{Set} \qquad \text{good}B : B_{\text{pre}} \to A_{\text{pre}} \to \text{Set}$$

to filter out the good elements.

$$(\text{good}B(b, a) \text{ inhabitated} \Leftrightarrow \text{“}b : B(a)\text{”})$$

# The general picture (cont.)

- We can then define

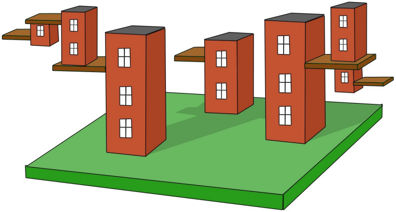
$$[\![A]\!] :\equiv (\Sigma a : A_{\mathrm{pre}})\,\mathrm{good}A(a)$$

$$[\![B]\!](a, ag) :\equiv (\Sigma b : B_{\mathrm{pre}})\,\mathrm{good}B(b, a) \quad .$$

- Need to show that the introduction and elimination rules hold.
- For the elimination rules, things become a lot simpler if we work in extensional type theory.

# The specific picture

- Formally, all this is done for an arbitrary code $\gamma$ representing inductive-inductively defined $A_\gamma$, $B_\gamma$.
- We map such codes to codes for IID.
- In this talk, we will illustrate the construction on a specific example, namely the platforms and buildings.



# Pre-platforms

For the "first approximation", we simply drop all index information:

$$\frac{}{\mathsf{ground} : \mathsf{Platform}} \qquad \frac{p : \mathsf{Platform} \qquad b : \mathsf{Building}(p)}{\mathsf{extension}(p, b) : \mathsf{Platform}}$$

becomes

$$\frac{}{\mathsf{ground}_{\mathsf{pre}} : \mathsf{Platform}_{\mathsf{pre}}} \qquad \frac{p : \mathsf{Platform}_{\mathsf{pre}} \qquad b : \mathsf{Building}_{\mathsf{pre}}}{\mathsf{ext}_{\mathsf{pre}}(p, b) : \mathsf{Platform}_{\mathsf{pre}}}$$

# Pre-buildings

$$\frac{p : \mathsf{Platform}}{\mathsf{onTop}(p) : \mathsf{Building}(p)}$$

becomes

$$\frac{p : \mathsf{Platform}_{\mathsf{pre}}}{\mathsf{onTop}_{\mathsf{pre}}(p) : \mathsf{Building}_{\mathsf{pre}}}$$

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$$\frac{p : \mathsf{Platform}}{\mathsf{onTop}(p) : \mathsf{Building}(p)}$$

$$\frac{p : \mathsf{Platform} \qquad b : \mathsf{Building}(p)}{\mathsf{hangingUnder}(p, b) : \mathsf{Building}(\mathsf{extension}(p, b))}$$

becomes

$$\frac{p : \mathsf{Platform}_{\mathsf{pre}}}{\mathsf{onTop}_{\mathsf{pre}}(p) : \mathsf{Building}_{\mathsf{pre}}}$$

$$\frac{p : \mathsf{Platform}_{\mathsf{pre}} \qquad b : \mathsf{Building}_{\mathsf{pre}}}{\mathsf{HU}_{\mathsf{pre}}(p, b) : \mathsf{Building}_{\mathsf{pre}}}$$

# Good platforms

Instead, the indices come back in the goodness predicates.

$$\frac{}{\mathsf{ground} : \mathsf{Platform}}$$

becomes

$$\frac{}{\mathsf{ground}_{\mathsf{good}} : \mathsf{goodPlatform}(\mathsf{ground}_{\mathsf{pre}})}$$

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$$\frac{}{\text{ground} : \text{Platform}} \qquad \frac{p : \text{Platform} \qquad b : \text{Building}(p)}{\text{extension}(p, b) : \text{Platform}}$$

becomes

$$\frac{}{\text{ground}_{\text{good}} : \text{goodPlatform}(\text{ground}_{\text{pre}})}$$

$$\frac{\begin{array}{cc} p : \text{Platform}_{\text{pre}} & b : \text{Building}_{\text{pre}} \\ gp : \text{goodPlatform}(p) & gb : \text{goodBuilding}(b, p) \end{array}}{\text{ext}_{\text{good}}(p, gp, b, gb) : \text{goodPlatform}(\text{ext}_{\text{pre}}(p, b))}$$

# Good buildings

$$\frac{p : \mathsf{Platform}}{\mathsf{onTop}(p) : \mathsf{Building}(p)}$$

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$$\frac{p : \mathsf{Platform}_{\mathsf{pre}} \qquad gp : \mathsf{goodPlatform}(p)}{\mathsf{onTop}_{\mathsf{good}}(p, gp) : \mathsf{goodBuilding}(\mathsf{onTop}_{\mathsf{pre}}(p), p)}$$

# Good buildings

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# Formation rules

$$[\![\mathsf{Platform}]\!] := (\Sigma p : \mathsf{Platform}_{\mathsf{pre}})\, \mathsf{goodPlatform}(p)$$

$$[\![\mathsf{Building}]\!](\langle p, gp \rangle) := (\Sigma b : \mathsf{Building}_{\mathsf{pre}})\, \mathsf{goodBuilding}(b, p)$$

## Taking a step back

Given an ind.-ind. definition

$$\mathsf{Platform} : \mathsf{Set} \qquad \mathsf{Building} : \mathsf{Platform} \to \mathsf{Set}$$

we have defined

$$[\![\mathsf{Platform}]\!] : \mathsf{Set} \qquad [\![\mathsf{Building}]\!] : [\![\mathsf{Platform}]\!] \to \mathsf{Set}$$

using only IID.

Must now show that original intro. and elim. rules are definable.

# Introduction rules

$$
\begin{aligned}
[\![\mathsf{ground}]\!] &: [\![\mathsf{Platform}]\!] \\
[\![\mathsf{ground}]\!] &= \{?\} \\
[\![\mathsf{extension}]\!] &: (x : [\![\mathsf{Platform}]\!]) \to [\![\mathsf{Building}]\!](x) \to [\![\mathsf{Platform}]\!] \\
[\![\mathsf{extension}]\!](\langle p, gp\rangle, \langle b, gb\rangle) &= \{?\} \\
[\![\mathsf{onTop}]\!] &: (x : [\![\mathsf{Platform}]\!]) \to [\![\mathsf{Building}]\!](x) \\
[\![\mathsf{onTop}]\!](\langle p, gp\rangle) &= \{?\} \\
[\![\mathsf{hangingUnder}]\!] &: (x : [\![\mathsf{Platform}]\!]) \to (y : [\![\mathsf{Building}]\!](x)) \\
&\qquad\qquad\qquad\qquad \to [\![\mathsf{Building}]\!]([\![\mathsf{extension}]\!](x, y)) \\
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# Introduction rules

$$[\![\text{ground}]\!] : [\![\text{Platform}]\!]$$

$$[\![\text{ground}]\!] = \langle \{?_0 : \text{Platform}_{\text{pre}}\}, \{?_1 : \text{goodPlatform}(?_0)\} \rangle$$

$$[\![\text{extension}]\!] : (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \to [\![\text{Platform}]\!]$$

$$[\![\text{extension}]\!](\langle p, gp\rangle, \langle b, gb\rangle) = \{?\}$$

$$[\![\text{onTop}]\!] : (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x)$$

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$x)$

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$[\![$han … $\text{ng}]\!](x))$

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$$[\![\text{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle) =$$

$$\langle \text{ext}_{\text{pre}}(\{?_2 : \text{Platform}_{\text{pre}}\}, \{?_3 : \text{Building}_{\text{pre}}\}), \{?_1 : \text{goodPlatform}(?_0)\} \rangle$$

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$$\llbracket \text{extension} \rrbracket(\langle p, gp \rangle, \langle b, gb \rangle) =$$

$$\langle \text{ext}_{\text{pre}}(p, \{?_3 : \text{Building}_{\text{pre}}\}), \{?_1 : \text{goodPlatform}(?_0)\} \rangle$$

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$$[\![\text{extension}]\!] : (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \to [\![\text{Platform}]\!]$$

$$[\![\text{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle) =$$

$$\langle \text{ext}_{\text{pre}}(p, b), \{?_1 : \text{goodPlatform}(\text{ext}_{\text{pre}}(p, b))\} \rangle$$

$$[\![\text{onTop}]\!] : (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x)$$

$$[\![\text{onTop}]\!](\langle p, gp \rangle) = \{?\}$$

$$\frac{p : \text{Platform}_{\text{pre}} \qquad b : \text{Building}_{\text{pre}} \qquad gp : \text{goodPlatform}(p) \qquad gb : \text{goodBuilding}(b, p)}{\text{ext}_{\text{good}}(p, gp, b, gb) : \text{goodPlatform}(\text{ext}_{\text{pre}}(p, b))}$$

# Introduction rules

$$[\![\text{ground}]\!] : [\![\text{Platform}]\!]$$
$$[\![\text{ground}]\!] = \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle$$
$$[\![\text{extension}]\!] : (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \to [\![\text{Platform}]\!]$$
$$[\![\text{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle) = \langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle$$
$$[\![\text{onTop}]\!] : (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x)$$
$$[\![\text{onTop}]\!](\langle p, gp \rangle) = \{?\}$$
$$[\![\text{hangingUnder}]\!] : (x : [\![\text{Platform}]\!]) \to (y : [\![\text{Building}]\!](x)) \to [\![\text{Building}]\!]([\![\text{extension}]\!](x, y))$$
$$[\![\text{hangingUnder}]\!](\langle p, gp \rangle, \langle b, gb \rangle) = \{?\}$$

# Introduction rules

$$
\begin{aligned}
[\![\text{ground}]\!] &: [\![\text{Platform}]\!] \\
[\![\text{ground}]\!] &= \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle \\
[\![\text{extension}]\!] &: (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \to [\![\text{Platform}]\!] \\
[\![\text{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle) &= \\
&\langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle \\
[\![\text{onTop}]\!] &: (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \\
[\![\text{onTop}]\!](\langle p, gp \rangle) &= \{?\} \\
[\![\text{hangingUnder}]\!] &: (x : [\![\text{Platform}]\!]) \to (y : [\![\text{Building}]\!](x)) \\
&\to [\![\text{Building}]\!]([\![\text{extension}]\!](x, y)) \\
[\![\text{hangingUnder}]\!](\langle p, gp \rangle, \langle b, gb \rangle) &= \{?\}
\end{aligned}
$$

# Introduction rules

$$
\begin{aligned}
[\![\text{ground}]\!] &: [\![\text{Platform}]\!] \\
[\![\text{ground}]\!] &= \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle \\
[\![\text{extension}]\!] &: (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \to [\![\text{Platform}]\!] \\
[\![\text{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle) &= \\
&\langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle \\
[\![\text{onTop}]\!] &: (x : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](x) \\
[\![\text{onTop}]\!](\langle p, gp \rangle) &= \langle \text{onTop}_{\text{pre}}(p), \text{onTop}_{\text{good}}(p, gp) \rangle \\
[\![\text{hangingUnder}]\!] &: (x : [\![\text{Platform}]\!]) \to (y : [\![\text{Building}]\!](x)) \\
&\qquad \to [\![\text{Building}]\!]([\![\text{extension}]\!](x, y)) \\
[\![\text{hangingUnder}]\!](\langle p, gp \rangle, \langle b, gb \rangle) &= \{?\}
\end{aligned}
$$

# Introduction rules

$$
\begin{aligned}
[\![\mathsf{ground}]\!] &: [\![\mathsf{Platform}]\!] \\
[\![\mathsf{ground}]\!] &= \langle \mathsf{ground}_{\mathsf{pre}}, \mathsf{ground}_{\mathsf{good}} \rangle \\
[\![\mathsf{extension}]\!] &: (x : [\![\mathsf{Platform}]\!]) \to [\![\mathsf{Building}]\!](x) \to [\![\mathsf{Platform}]\!] \\
[\![\mathsf{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle) &= \\
&\langle \mathsf{ext}_{\mathsf{pre}}(p, b), \mathsf{ext}_{\mathsf{good}}(p, gp, b, gb) \rangle \\
[\![\mathsf{onTop}]\!] &: (x : [\![\mathsf{Platform}]\!]) \to [\![\mathsf{Building}]\!](x) \\
[\![\mathsf{onTop}]\!](\langle p, gp \rangle) &= \langle \mathsf{onTop}_{\mathsf{pre}}(p), \mathsf{onTop}_{\mathsf{good}}(p, gp) \rangle \\
[\![\mathsf{hangingUnder}]\!] &: (x : [\![\mathsf{Platform}]\!]) \to (y : [\![\mathsf{Building}]\!](x)) \\
&\qquad \to [\![\mathsf{Building}]\!]([\![\mathsf{extension}]\!](x, y)) \\
[\![\mathsf{hangingUnder}]\!](\langle p, gp \rangle, \langle b, gb \rangle) &= \langle \mathsf{HU}_{\mathsf{pre}}(p, b), \mathsf{HU}_{\mathsf{good}}(p, gp, b, gb) \rangle
\end{aligned}
$$

# Elimination rules

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Platform}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \\
& (P' : (p : [\![\mathrm{Platform}]\!]) \to [\![\mathrm{Building}]\!](p) \to \mathrm{Set}) \to \\
& (\mathrm{base}_{\mathrm{Platform}} : P([\![\mathrm{ground}]\!])) \to \\
& (\mathrm{step}_{\mathrm{Platform}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathrm{extension}]\!](p, b))) \to \\
& (\mathrm{base}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to P'(p, [\![\mathrm{onTop}]\!](p))) \to \\
& (\mathrm{step}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathrm{extension}]\!](p, b), [\![\mathrm{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathrm{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Building}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \ldots \to \\
& \qquad (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P'(p, b)
\end{aligned}
$$

# Elimination rules

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Platform}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \\
& (P' : (p : [\![\mathrm{Platform}]\!]) \to [\![\mathrm{Building}]\!](p) \to \mathrm{Set}) \to \\
& (\mathrm{base}_{\mathrm{Platform}} : P([\![\mathrm{ground}]\!])) \to \\
& (\mathrm{step}_{\mathrm{Platform}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathrm{extension}]\!](p, b))) \to \\
& (\mathrm{base}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to P'(p, [\![\mathrm{onTop}]\!](p))) \to \\
& (\mathrm{step}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathrm{extension}]\!](p, b), [\![\mathrm{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathrm{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Building}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \ldots \to \\
& \qquad (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P'(p, b)
\end{aligned}
$$

# Elimination rules

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Platform}} :\ & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \\
& (P' : (p : [\![\mathrm{Platform}]\!]) \to [\![\mathrm{Building}]\!](p) \to \mathrm{Set}) \to \\
& (\mathrm{base}_{\mathrm{Platform}} : P([\![\mathrm{ground}]\!])) \to \\
& (\mathrm{step}_{\mathrm{Platform}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathrm{extension}]\!](p, b))) \to \\
& (\mathrm{base}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to P'(p, [\![\mathrm{onTop}]\!](p))) \to \\
& (\mathrm{step}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathrm{extension}]\!](p, b), [\![\mathrm{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathrm{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Building}} :\ & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \ldots \to \\
& \qquad (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P'(p, b)
\end{aligned}
$$

# Elimination rules

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Platform}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \\
& (P' : (p : [\![\mathrm{Platform}]\!]) \to [\![\mathrm{Building}]\!](p) \to \mathrm{Set}) \to \\
& (\mathrm{base}_{\mathrm{Platform}} : P([\![\mathrm{ground}]\!])) \to \\
& (\mathrm{step}_{\mathrm{Platform}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathrm{extension}]\!](p, b))) \to \\
& (\mathrm{base}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to P'(p, [\![\mathrm{onTop}]\!](p))) \to \\
& (\mathrm{step}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathrm{extension}]\!](p, b), [\![\mathrm{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathrm{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Building}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \ldots \to \\
& \qquad (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P'(p, b)
\end{aligned}
$$

# $\mathrm{elim}_{\mathrm{Platform}}$

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Platform}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \\
& (P' : (p : [\![\mathrm{Platform}]\!]) \to [\![\mathrm{Building}]\!](p) \to \mathrm{Set}) \to \\
& (\mathrm{base}_{\mathrm{Platform}} : P([\![\mathrm{ground}]\!])) \to \\
& (\mathrm{step}_{\mathrm{Platform}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathrm{extension}]\!](p, b))) \to \\
& (\mathrm{base}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to P'(p, [\![\mathrm{onTop}]\!](p))) \to \\
& (\mathrm{step}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathrm{extension}]\!](p, b), [\![\mathrm{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathrm{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\mathrm{elim}_{\mathrm{Platform}}(\ldots, x) = \{?_0 : P(x)\}
$$

# $\text{elim}_{\text{Platform}}$

$$
\begin{aligned}
\text{elim}_{\text{Platform}} : {} & (P : [\![\text{Platform}]\!] \to \text{Set}) \to \\
& (P' : (p : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](p) \to \text{Set}) \to \\
& (\text{base}_{\text{Platform}} : P([\![\text{ground}]\!])) \to \\
& (\text{step}_{\text{Platform}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\text{extension}]\!](p, b))) \to \\
& (\text{base}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to P'(p, [\![\text{onTop}]\!](p))) \to \\
& (\text{step}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\text{extension}]\!](p, b), [\![\text{hangingUnder}]\!](p, b))) \to \\
& \qquad (p : [\![\text{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$\text{elim}_{\text{Platform}}(\ldots, \langle p', gp \rangle) = \{?_0 : P(\langle p', gp \rangle)\}$$

# $\text{elim}_{\text{Platform}}$

$$
\begin{aligned}
\text{elim}_{\text{Platform}} : {} & (P : [\![\text{Platform}]\!] \to \text{Set}) \to \\
& (P' : (p : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](p) \to \text{Set}) \to \\
& (\text{base}_{\text{Platform}} : P([\![\text{ground}]\!])) \to \\
& (\text{step}_{\text{Platform}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\text{extension}]\!](p, b))) \to \\
& (\text{base}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to P'(p, [\![\text{onTop}]\!](p))) \to \\
& (\text{step}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\text{extension}]\!](p, b), [\![\text{hangingUnder}]\!](p, b))) \to \\
& \qquad (p : [\![\text{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\text{elim}_{\text{Platform}}(\ldots, \langle p', \text{ground}_{\text{good}} \rangle) = \{?_0 : P(\langle p', \text{ground}_{\text{good}} \rangle)\}
$$

$$
\text{elim}_{\text{Platform}}(\ldots, \langle p', \text{ext}_{\text{good}}(p, gp, b, gb) \rangle) = \{?_1 : P(\langle p', \text{ext}_{\text{good}}(p, b, gp, gb) \rangle)\}
$$

# $\mathsf{elim}_{\mathsf{Platform}}$

$$
\begin{aligned}
\mathsf{elim}_{\mathsf{Platform}} : {} & (P : [\![\mathsf{Platform}]\!] \to \mathsf{Set}) \to \\
& (P' : (p : [\![\mathsf{Platform}]\!]) \to [\![\mathsf{Building}]\!](p) \to \mathsf{Set}) \to \\
& (\mathsf{base}_{\mathsf{Platform}} : P([\![\mathsf{ground}]\!])) \to \\
& (\mathsf{step}_{\mathsf{Platform}} : (p : [\![\mathsf{Platform}]\!]) \to (b : [\![\mathsf{Building}]\!](p)) \to P(p) \\
& \quad \to P'(p, b) \to P([\![\mathsf{extension}]\!](p, b))) \to \\
& (\mathsf{base}_{\mathsf{Building}} : (p : [\![\mathsf{Platform}]\!]) \to P'(p, [\![\mathsf{onTop}]\!](p))) \to \\
& (\mathsf{step}_{\mathsf{Building}} : (p : [\![\mathsf{Platform}]\!]) \to (b : [\![\mathsf{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathsf{extension}]\!](p, b), [\![\mathsf{hangingUnder}]\!](p, b))) \to \\
& \qquad (p : [\![\mathsf{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$\mathsf{elim}_{\mathsf{Platform}}(\ldots, \langle \mathsf{ground}_{\mathsf{pre}}, \mathsf{ground}_{\mathsf{good}} \rangle) = \{?_0 : P(\langle \mathsf{ground}_{\mathsf{pre}}, \mathsf{ground}_{\mathsf{good}} \rangle$$

$$\mathsf{elim}_{\mathsf{Platform}}(\ldots, \langle \mathsf{ext}_{\mathsf{pre}}(p, b), \mathsf{ext}_{\mathsf{good}}(p, gp, b, gb) \rangle) =$$

$$\{?_1 : P(\langle \mathsf{ext}_{\mathsf{pre}}(p, b), \mathsf{ext}_{\mathsf{good}}(p, b, gp, gb) \rangle)\}$$

# $\text{elim}_{\text{Platform}}$

$$
\begin{aligned}
\text{elim}_{\text{Platform}} : {} & (P : [\![\text{Platform}]\!] \to \text{Set}) \to \\
& (P' : (p : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](p) \to \text{Set}) \to \\
& (\text{base}_{\text{Platform}} : P([\![\text{ground}]\!])) \to \\
& (\text{step}_{\text{Platform}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\text{extension}]\!](p, b))) \to \\
& (\text{base}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to P'(p, [\![\text{onTop}]\!](p))) \to \\
& (\text{step}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \qquad \to P'([\![\text{extension}]\!](p, b), [\![\text{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\text{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$\text{elim}_{\text{Platform}}(\ldots, \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle) = \{?_0 : P([\![\text{ground}]\!])\}$$

$$\text{elim}_{\text{Platform}}(\ldots, \langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle) = \{?_1 : P(\langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, b, gp, gb) \rangle)\}$$

# $\text{elim}_{\text{Platform}}$

$$
\begin{aligned}
\text{elim}_{\text{Platform}} : {} & (P : [\![\text{Platform}]\!] \to \text{Set}) \to \\
& (P' : (p : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](p) \to \text{Set}) \to \\
& (\text{base}_{\text{Platform}} : P([\![\text{ground}]\!])) \to \\
& (\text{step}_{\text{Platform}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\text{extension}]\!](p, b))) \to \\
& (\text{base}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to P'(p, [\![\text{onTop}]\!](p))) \to \\
& (\text{step}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \qquad \to P'([\![\text{extension}]\!](p, b), [\![\text{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\text{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\begin{aligned}
\text{elim}_{\text{Platform}}(\ldots, \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle) & = \text{base}_{\text{Platform}} \\
\text{elim}_{\text{Platform}}(\ldots, \langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle) & = \\
& \{?_1 : P(\langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, b, gp, gb) \rangle)\}
\end{aligned}
$$

# $\text{elim}_{\text{Platform}}$

$$
\begin{aligned}
\text{elim}_{\text{Platform}} : \ & (P : [\![\text{Platform}]\!] \to \text{Set}) \to \\
& (P' : (p : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](p) \to \text{Set}) \to \\
& (\text{base}_{\text{Platform}} : P([\![\text{ground}]\!])) \to \\
& (\text{step}_{\text{Platform}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\text{extension}]\!](p, b))) \to \\
& (\text{base}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to P'(p, [\![\text{onTop}]\!](p))) \to \\
& (\text{step}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \qquad \to P'([\![\text{extension}]\!](p, b), [\![\text{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\text{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\begin{aligned}
\text{elim}_{\text{Platform}}(\ldots, \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle) &= \text{base}_{\text{Platform}} \\
\text{elim}_{\text{Platform}}(\ldots, \langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle) &= \\
& \{?_1 : P([\![\text{extension}]\!](\langle p, gp \rangle, \langle b, gb \rangle))\}
\end{aligned}
$$

# $\mathrm{elim}_{\mathrm{Platform}}$

$$
\begin{aligned}
\mathrm{elim}_{\mathrm{Platform}} : {} & (P : [\![\mathrm{Platform}]\!] \to \mathrm{Set}) \to \\
& (P' : (p : [\![\mathrm{Platform}]\!]) \to [\![\mathrm{Building}]\!](p) \to \mathrm{Set}) \to \\
& (\mathrm{base}_{\mathrm{Platform}} : P([\![\mathrm{ground}]\!])) \to \\
& (\mathrm{step}_{\mathrm{Platform}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathrm{extension}]\!](p, b))) \to \\
& (\mathrm{base}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to P'(p, [\![\mathrm{onTop}]\!](p))) \to \\
& (\mathrm{step}_{\mathrm{Building}} : (p : [\![\mathrm{Platform}]\!]) \to (b : [\![\mathrm{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \quad \to P'([\![\mathrm{extension}]\!](p, b), [\![\mathrm{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathrm{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\mathrm{elim}_{\mathrm{Platform}}(\ldots, \langle \mathrm{ground}_{\mathrm{pre}}, \mathrm{ground}_{\mathrm{good}} \rangle) = \mathrm{base}_{\mathrm{Platform}}
$$

$$
\mathrm{elim}_{\mathrm{Platform}}(\ldots, \langle \mathrm{ext}_{\mathrm{pre}}(p, b), \mathrm{ext}_{\mathrm{good}}(p, gp, b, gb) \rangle) =
$$

$$
\mathrm{step}_{\mathrm{Platform}}(\langle p, gp \rangle, \langle b, gb \rangle, \{?_2 : P(\langle p, gp \rangle)\}, \{?_3 : P'(\langle p, gp \rangle, \langle b, gb \rangle)\})
$$

# $\mathsf{elim}_{\mathsf{Platform}}$

$$
\begin{aligned}
\mathsf{elim}_{\mathsf{Platform}} : {} & (P : [\![\mathsf{Platform}]\!] \to \mathsf{Set}) \to \\
& (P' : (p : [\![\mathsf{Platform}]\!]) \to [\![\mathsf{Building}]\!](p) \to \mathsf{Set}) \to \\
& (\mathsf{base}_{\mathsf{Platform}} : P([\![\mathsf{ground}]\!])) \to \\
& (\mathsf{step}_{\mathsf{Platform}} : (p : [\![\mathsf{Platform}]\!]) \to (b : [\![\mathsf{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\mathsf{extension}]\!](p, b))) \to \\
& (\mathsf{base}_{\mathsf{Building}} : (p : [\![\mathsf{Platform}]\!]) \to P'(p, [\![\mathsf{onTop}]\!](p))) \to \\
& (\mathsf{step}_{\mathsf{Building}} : (p : [\![\mathsf{Platform}]\!]) \to (b : [\![\mathsf{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \qquad \to P'([\![\mathsf{extension}]\!](p, b), [\![\mathsf{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\mathsf{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\mathsf{elim}_{\mathsf{Platform}}(\ldots, \langle \mathsf{ground}_{\mathsf{pre}}, \mathsf{ground}_{\mathsf{good}} \rangle) = \mathsf{base}_{\mathsf{Platform}}
$$

$$
\mathsf{elim}_{\mathsf{Platform}}(\ldots, \langle \mathsf{ext}_{\mathsf{pre}}(p, b), \mathsf{ext}_{\mathsf{good}}(p, gp, b, gb) \rangle) =
$$

$$
\mathsf{step}_{\mathsf{Platform}}(\langle p, gp \rangle, \langle b, gb \rangle, \mathsf{elim}_{\mathsf{Platform}}(\ldots, \langle p, gp \rangle), \{?_3 : P'(\langle p, gp \rangle, \langle b, gb \rangle)
$$

# $\text{elim}_{\text{Platform}}$

$$
\begin{aligned}
\text{elim}_{\text{Platform}} : \ & (P : [\![\text{Platform}]\!] \to \text{Set}) \to \\
& (P' : (p : [\![\text{Platform}]\!]) \to [\![\text{Building}]\!](p) \to \text{Set}) \to \\
& (\text{base}_{\text{Platform}} : P([\![\text{ground}]\!])) \to \\
& (\text{step}_{\text{Platform}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \\
& \qquad \to P'(p, b) \to P([\![\text{extension}]\!](p, b))) \to \\
& (\text{base}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to P'(p, [\![\text{onTop}]\!](p))) \to \\
& (\text{step}_{\text{Building}} : (p : [\![\text{Platform}]\!]) \to (b : [\![\text{Building}]\!](p)) \to P(p) \to P'(p, b) \\
& \qquad \to P'([\![\text{extension}]\!](p, b), [\![\text{hangingUnder}]\!](p, b))) \to \\
& \qquad\qquad (p : [\![\text{Platform}]\!]) \to P(p)
\end{aligned}
$$

$$
\text{elim}_{\text{Platform}}(\ldots, \langle \text{ground}_{\text{pre}}, \text{ground}_{\text{good}} \rangle) = \text{base}_{\text{Platform}}
$$

$$
\text{elim}_{\text{Platform}}(\ldots, \langle \text{ext}_{\text{pre}}(p, b), \text{ext}_{\text{good}}(p, gp, b, gb) \rangle) =
$$

$$
\text{step}_{\text{Platform}}(\langle p, gp \rangle, \langle b, gb \rangle, \text{elim}_{\text{Platform}}(\ldots, \langle p, gp \rangle), \text{elim}_{\text{Building}}(\ldots))
$$

# $\mathrm{elim}_{\mathrm{Building}}$

For $\mathrm{elim}_{\mathrm{Building}}$, the story is similar, but we also need to prove

**Lemma**

*Let* $\Gamma : \mathit{Platform}_{\mathit{pre}}$. *For all* $\Gamma g, \Gamma g' : \mathit{goodPlatform}(\Gamma)$,

$$\Gamma g =_{\mathit{goodPlatform}(\Gamma)} \Gamma g'$$

(follows from elim. rules for goodPlatform)

With extensional type theory and its equality reflection, we can then define $\mathrm{elim}_{\mathrm{Building}}$ such that the computation rules hold.

# Results

$$\textsc{IndInd} \leq \textsc{IndInd}^{\mathsf{ext}} \leq \mathrm{IID}^{\mathsf{ext}}$$

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Still work in progress,
but nearly there

$$\textsc{IndInd} \;\leq\; \textsc{IndInd}^{\mathsf{ext}} \;\text{“}\!\leq\!\text{”}\; \mathrm{IID}^{\mathsf{ext}}$$

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Still work in progress, but nearly there

$$\mathrm{IID} \;\leq\; \textsc{IndInd} \;\leq\; \textsc{IndInd}^{\mathsf{ext}} \;\text{``}\leq\text{''}\; \mathrm{IID}^{\mathsf{ext}}$$

Let $A$ be isomorphic copy of $I$ (constructor $\mathrm{intro}_{\mathrm{A}} : I \to A$)

# Results

Still work in progress, but nearly there

$$\mathrm{IID} \;\leq\; \textsc{IndInd} \;\leq\; \textsc{IndInd}^{\mathsf{ext}} \;\text{“}\!\leq\!\text{”}\; \mathrm{IID}^{\mathsf{ext}}$$

Let $A$ be isomorphic copy of $I$ (constructor $\mathsf{intro}_{\mathrm{A}} : I \to A$)

## When it comes to well orderings

Likely that $|\mathrm{IID}| = |\mathrm{IID}^{\mathsf{ext}}|$, so that $|\textsc{IndInd}| = |\mathrm{IID}|$.

# Results

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Thanks!