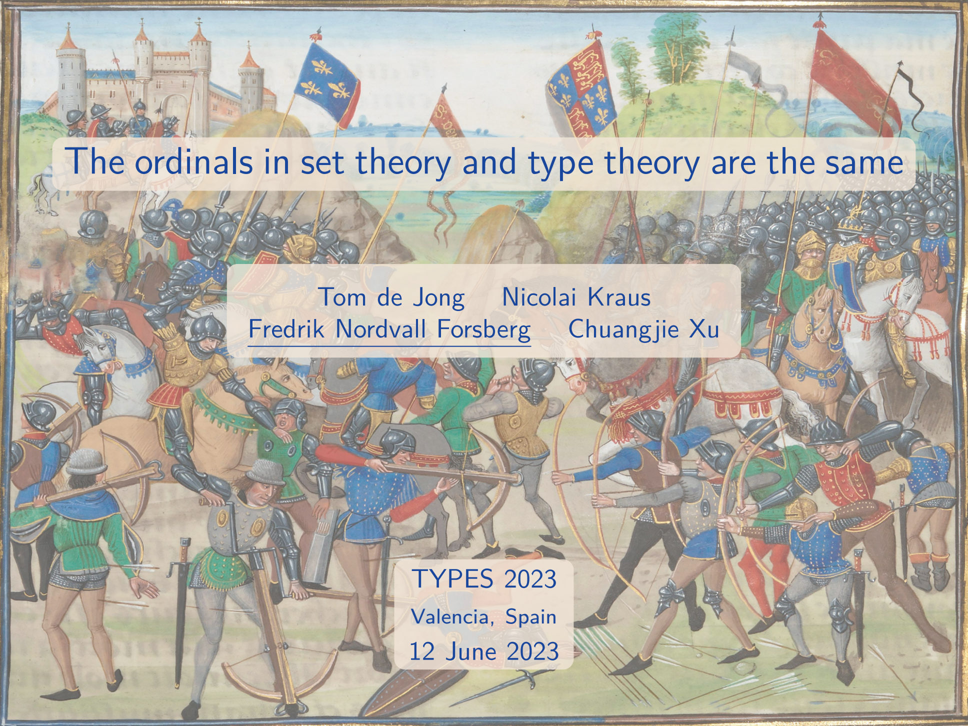

The ordinals in set theory and type theory are the same

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The usefulness of ordinals

- Give semantics to inductive data types.
  Construct initial algebras by transfinite iteration.

- Justify recursion and termination of programs.
  Construct a strictly decreasing measure.

- Determine the proof-theoretic strength of a formal system T.
  Find least ordinal α such that T cannot prove α well ordered.

- Interesting structure in their own right.
  For example with rich theory of arithmetic.

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Want to do this internally to our type theory/topos/programming language
$\implies$ Want a constructive theory of ordinals.

Ordinals in set theory

There are many classically equivalent notions of ordinals in set theory. The following is constructively acceptable [Powell 1975, Aczel–Rathjen 2010].

Def. A set x is transitive if $z \in y$ and $y \in x$ implies $z \in x$.

Def. A set-theoretic ordinal is a transitive set whose elements are all transitive.

Examples $0 := \emptyset$, $1 := \{\emptyset\}$, $2 := \{\emptyset, \{\emptyset\}\}$, ..., $\mathbb{N} := \{0, 1, 2, \ldots\}$, ... are all set-theoretic ordinals.

Ordinals in homotopy type theory

In type theory, the statement "$z : y$ and $y : x$ implies $z : x$" makes no sense. The HoTT book [§ 10.3] instead defines ordinals as follows.

Def. A (type-theoretic) ordinal is a type X with a prop-valued binary relation $<$ that is transitive, extensional and wellfounded.

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Wellfoundedness is defined in terms of accessibility, but is equivalent to transfinite induction: for every $P : X \to \mathcal{U}$, we have $\Pi(x : X).P(x)$ as soon as $\Pi(x : X).(\Pi(y : X).(y < x \to P(y))) \to P(x)$.

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Thm. (HoTT Book 10.3.20) The type Ord is itself a (large) type-theoretic ordinal with relation $\prec$ given by

$$\begin{aligned} \alpha \prec \beta &\iff \alpha \text{ is an initial segment of } \beta \\ &\iff \Sigma(y : \beta).(\alpha = \beta \downarrow y) \end{aligned}$$

where we write $\beta \downarrow y$ for the (sub)ordinal $\Sigma(x : \beta).(x < y)$.

That is, $\prec$ is prop-valued, transitive, extensional and wellfounded.

The cumulative hierarchy in HoTT

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The type $\mathbb{V}$ is a higher inductive type with point constructor

$$\mathbb{V}\text{-set} : (\Sigma(A : \mathcal{U}).(A \to \mathbb{V})) \to \mathbb{V}$$

quotiented by bisimilarity: $\mathbb{V}\text{-set}(A, f)$ and $\mathbb{V}\text{-set}(B, g)$ are identified exactly when f and g have the same image.

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For example, the empty set is represented by $\mathbb{V}\text{-set}(\mathbf{0}, \mathbf{0}\text{-rec})$, and if $x : \mathbb{V}$, then the singleton $\{x\}$ is represented by $\mathbb{V}\text{-set}(\mathbf{1}, \lambda(u : \mathbf{1}).x)$.

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This is a refinement of Aczel's 1978 model of CZF in type theory (see also Gylterud [2018]).

The ordinal of set-theoretic ordinals

Def. We define set-membership $\in : \mathbb{V} \to \mathbb{V} \to \mathsf{Prop}$ by

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Using $\in$, we define the subtype $\mathbb{V}_{\mathsf{ord}}$ of $\mathbb{V}$ of set-theoretic ordinals in HoTT:

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The cumulative hierarchy $\mathbb{V}$ validates the axioms of $\in$-extensionality and $\in$-induction. Since $\mathbb{V}_{\mathsf{ord}}$ is restricted to hereditarily transitive sets, we get:

Thm. $(\mathbb{V}_{\mathsf{ord}}, \in)$ is a type-theoretic ordinal.

Set-theoretic and type-theoretic ordinals coincide

Thm. $(\mathbb{V}_{\mathsf{ord}}, \in)$ and $(\mathsf{Ord}, \prec)$ are equivalent as ordinals. Hence, by univalence, they are equal.

Thus, in HoTT,

set-theoretic and type-theoretic ordinals coincide.

From type-theoretic ordinals to set-theoretic ordinals

Define $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$ by transfinite recursion:

$$\Phi(\alpha) :\equiv \mathbb{V}\text{-set}(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)).$$

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$$\Phi(\alpha) :\equiv \mathbb{V}\text{-set}(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)).$$

This is well-defined, because $(\alpha \downarrow a) \prec \alpha$, and the fact that $\prec$ on Ord is wellfounded.

From set-theoretic ordinals to type-theoretic ordinals

The map $\Psi : \mathbb{V}_{\mathsf{ord}} \to \mathsf{Ord}$ is the rank function:

$$\Psi(\mathbb{V}\text{-set}(A, f)) :\equiv \bigvee_{a:A} (\Psi(f(a)) + \mathbf{1}),$$

where $\bigvee$ denotes the supremum of ordinals, which exists for any small family of ordinals [de Jong–Escardó 2023].

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It is possible to give nonrecursive descriptions of the rank:

$$\Psi(x) \simeq \Sigma(y : \mathbb{V}).y \in x \quad \text{and} \quad \Psi(\mathbb{V}\text{-set}(A, f)) = A/{\sim},$$

where $a \sim b \iff f(a) = f(b)$. (But be careful about size.)

Set-theoretic and type-theoretic ordinals coincide

Thm. The type-theoretic ordinals $(\mathbb{V}_{\mathsf{ord}}, \in)$ and $(\mathsf{Ord}, \prec)$ are equivalent.

Proof sketch The maps $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$ and $\Psi : \mathbb{V}_{\mathsf{ord}} \to \mathsf{Ord}$ give an isomorphism of ordinals. In particular,

$$\alpha \prec \beta \iff \Phi(\alpha) \in \Phi(\beta) \quad \text{and} \quad x \in y \iff \Psi(x) \prec \Psi(y). \quad \square$$

Capturing all of the cumulative hierarchy

Can we realize *all* of $\mathbb{V}$ as a type of ordered structures?

That is, can we find a type making the square

$$\begin{array}{ccc} \mathbb{V}_{\mathsf{ord}} & \xrightarrow{\simeq} & \mathsf{Ord} \\ \big\downarrow & & \big\downarrow \\ \mathbb{V} & \xrightarrow{\simeq} & ? \end{array}$$

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An initial first attempt may be to simply drop transitivity, i.e., to take

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This does not work for cardinality reasons: there are more subsets of $\{\emptyset, \{\emptyset\}\}$ than extensional wellfounded relations embedding into $0 < 1$.

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Similar ideas of encoding sets as wellfounded structures can be found in Osius [1974], Aczel [1977, 1988], Taylor [1996], and Adamek et al. [2013].

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Def. We write $\mathsf{MEWO}_{\mathsf{cov}}$ for the type of covered marked extensional wellfounded order relations.

Every ordinal can be equipped with the trivial covering by marking everything (and forgetting transitivity). Hence Ord embeds into $\mathsf{MEWO}_{\mathsf{cov}}$.

Sets and covered marked extensional wellfounded relations are the same

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To show $\mathbb{V} = \mathsf{MEWO}_{\mathsf{cov}}$, we construct a mewo of mewos, and show that $\mathbb{V}$ and $\mathsf{MEWO}_{\mathsf{cov}}$ are equivalent as mewos, by generalising the constructions for $\mathbb{V}_{\mathsf{ord}}$ and Ord. (Coveredness crucial for well-definedness of mewo version of "$+\mathbf{1}$".)

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In particular, this gives a "non-inductive" presentation of the cumulative hierarchy $\mathbb{V}$.

Summary

In HoTT, the set-theoretic ordinals in $\mathbb{V}$ coincide with the type-theoretic ordinals.

By generalising from type-theoretic ordinals to covered mewos, we capture all sets in $\mathbb{V}$.

Question: Can we similarly capture non-wellfounded sets as certain graphs in HoTT?

Set-Theoretic and Type-Theoretic Ordinals Coincide.
Tom de Jong, Nicolai Kraus, Fredrik Nordvall Forsberg and Chuangjie Xu. arXiv:2301.10696. To appear at *LICS'23*.

Full Agda formalisation.
Building on Escardó's TypeTopology, and the agda/cubical library.
https://tdejong.com/agda-html/st-tt-ordinals/

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