Generalized Decidability via Brouwer Trees

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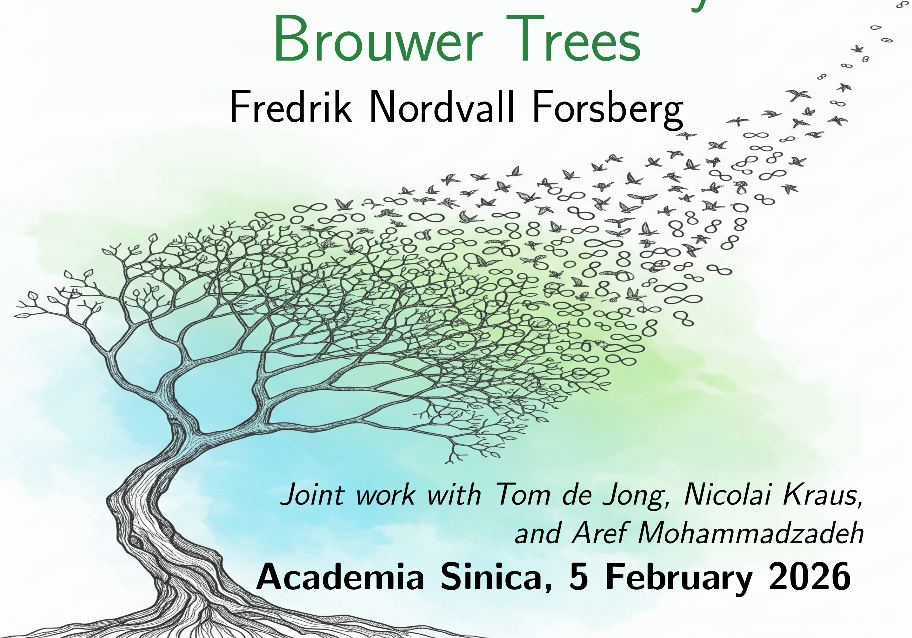

Joint work with Tom de Jong, Nicolai Kraus,
and Aref Mohammadzadeh

Academia Sinica, 5 February 2026

Decidability

Definition (classical): A proposition P is *decidable* if there is an always halting Turing machine T, such that T returns 1 iff P holds.

Examples:

- True and false propositions.
- The integer n is prime.
- There exists a satisfying assignment for the Boolean formula φ.

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- There exists a satisfying assignment for the Boolean formula φ.

Definition (synthetic/internal): A proposition P is *decidable* if we can prove $P \vee \neg P$.

Semidecidability

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Examples:

- Decidable propositions.
- Hilbert's 10^{th} Problem: does a given diophantine polynomial equation $P(x_1, \ldots, x_n) = 0$ have an integer solution?
- Given n, is there a twin prime pair above n, i.e., is there $p > n$ such that p and $p + 2$ are prime?

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Definition (synthetic/internal) [Rosolini 1986, Bauer 2006]: A proposition P is *semidecidable* if there exists $s : \mathbb{N} \to \text{Bool}$ such that

$$P \leftrightarrow \exists\,(n : \mathbb{N}).\, s\, n = \text{true}.$$

Beyond semidecidability

Many problems are not decidable or semidecidable.

Examples:

- Twin Prime Conjecture: there exist infinitely many prime numbers p such that $p + 2$ is also prime.
- ABC Conjecture: For every positive real number $\varepsilon > 0$, there exists only a finite number of triples (a, b, c) of coprime natural numbers such that $a + b = c$ and $c > \mathsf{rad}(abc)^{1+\varepsilon}$.

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Can we also capture such generalised notions of decidability internally? Intuitively, this would allow a more refined view on the "hardness" of a problem.

Ordinal decidability

Definition: Let α be an ordinal. A proposition P is said to be α-*decidable* if there exists an ordinal y such that

$$P \leftrightarrow \alpha \leq y.$$

Example: Semidecidable propositions are $\omega + 1$-decidable: given $s : \mathbb{N} \to \text{Bool}$ witnessing semidecidability, take $y := \text{limit } f$, where

$$f(n) = \begin{cases} \omega + n & \text{if } s(i) = \text{true for some } i \leq n \\ n & \text{otherwise} \end{cases}$$

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However: which notion of ordinals is appropriate for this?

Brouwer trees

Brw : Set is a quotient inductive-inductive type, with constructors [Kraus, Nordvall Forsberg, and Xu, 2023]:

$$\begin{aligned}
\mathsf{zero} &: \mathsf{Brw} \\
\mathsf{succ} &: \mathsf{Brw} \to \mathsf{Brw} \\
\mathsf{limit} &: (\mathbb{N} \xrightarrow{<} \mathsf{Brw}) \to \mathsf{Brw} \\
\mathsf{bisim} &: (f, g : \mathbb{N} \xrightarrow{<} \mathsf{Brw}) \to (f \approx g) \to \mathsf{limit}(f) = \mathsf{limit}(g)
\end{aligned}$$

Here $\mathbb{N} \xrightarrow{<} \mathsf{Brw}$ denotes the type of strictly increasing sequences.

The order relation on Brouwer trees

The constructors for the mutually defined relation $\leq : \mathsf{Brw} \to \mathsf{Brw} \to \mathsf{Prop}$ are:

$$\begin{aligned}
\leq\text{-zero} &: \mathsf{zero} \leq \alpha \\
\leq\text{-trans} &: \alpha \leq \beta \to \beta \leq \gamma \to \alpha \leq \gamma \\
\leq\text{-succ-mono} &: \alpha \leq \beta \to \mathsf{succ}\ \alpha \leq \mathsf{succ}\ \beta \\
\leq\text{-cocone} &: (k : \mathbb{N}) \to \alpha \leq f(k) \to \alpha \leq \mathsf{limit}\ f \\
\leq\text{-limiting} &: (\forall k. f(k) \leq \alpha) \to \mathsf{limit}\ f \leq \alpha
\end{aligned}$$

We define $\alpha < \beta := \mathsf{succ}\ \alpha \leq \beta$.

Recursion and induction on Brouwer trees

To define a function $f : \mathsf{Brw} \to X$, it suffices to give

- $f\,\mathsf{zero} = \boxed{?_0}$
- $f\,(\mathsf{succ}\ x) = \boxed{?_1}$ (assuming $f\,x$ already defined)
- $f\,(\mathsf{limit}\ g) = \boxed{?_3}$ (assuming $f\,(g\,k)$ already defined for $k : \mathbb{N}$)
- such that $f\,(\mathsf{limit}\ g) = f\,(\mathsf{limit}\ h)$ whenever $g \approx h$.

To prove $(\forall x : \mathsf{Brw}).\,P\,x$ for some $P : \mathsf{Brw} \to \mathsf{Prop}$, suffices to show

- $P\,\mathsf{zero}$
- $P\,x \implies P\,(\mathsf{succ}\ x)$
- $\big(\forall k. P\,(f\,k)\big) \implies P\,(\mathsf{limit}\ f)$

Properties of Brouwer trees

Lemma

(i) Every ordinal is either zero, or a successor, or a limit.

(ii) It is decidable whether a given ordinal α is finite (i.e. $\alpha < \omega$) or infinite (i.e. $\alpha \geq \omega$).

(iii) However being able to decide $\alpha \leq \beta$ or $\alpha = \beta$ in general is equivalent to LPO.

(iv) The following properties characterize the relation $\leq$:
  - succ $\alpha \leq$ succ β if and only if $\alpha \leq \beta$;
  - $\alpha <$ limit f if and only if there exists $n : \mathbb{N}$ such that $\alpha < f(n)$;
  - If limit $f \leq$ succ α then limit $f \leq \alpha$;
  - If $\alpha <$ limit f then succ $\alpha <$ limit f;
  - limit $f \leq$ limit g if and only if f is simulated by g.

Generalizing decidability

Let α be a Brouwer tree. A proposition P is said to be α-*decidable* if

$$\exists y : \mathsf{Brw}.\big(P \leftrightarrow \alpha \leq y\big).$$

Proposition: For any proposition P, the following statements are equivalent:

(i) P is decidable.

(ii) P is n-decidable for any finite ordinal $n > 0$.

(iii) P is ω-decidable.

Generalizing semidecidability

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We construct $s_y : \mathbb{N} \to \mathsf{Bool}$ by case distinction on y:

- $s_{\mathsf{zero}}(n) = \mathsf{false}$.
- $s_{\mathsf{succ}\ y'}(n) = \mathsf{false}$ if y' is finite, and $s_{\mathsf{succ}\ y'}(n) = \mathsf{true}$ if y' is infinite.
- $s_{\mathsf{limit}\ f}(n) = \begin{cases} \mathsf{false} & \text{if } f(n) \text{ is finite} \\ \mathsf{true} & \text{if } f(n) \text{ is infinite} \end{cases}$

Reduction to limit ordinals

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Definition: Define the "rounding-up function" $\Uparrow : \mathsf{Brw} \to \mathsf{Brw}^{zl}$ by

$$\begin{aligned} \Uparrow \mathsf{zero} &= \mathsf{zero} \\ \Uparrow (\mathsf{succ}\ x) &= x + \omega \quad (= \mathsf{limit}\,(\lambda n . x + n)) \\ \Uparrow (\mathsf{limit}\ f) &= \mathsf{limit}\ f \end{aligned}$$

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Similarly we can define a "rounding-down function" $\Downarrow : \mathsf{Brw} \to \mathsf{Brw}^{zl}$ and show that $\lambda \leq y \leftrightarrow \lambda \leq \Downarrow y$ when λ is zero or a limit.

Classification of α-decidability for low α

Corollary: For a proposition P, we have

$$\begin{aligned} P \text{ holds} &\leftrightarrow P \text{ is } (\omega \cdot 0)\text{-decidable} \\ P \text{ is decidable} &\leftrightarrow P \text{ is } (\omega \cdot 1)\text{-decidable} \\ P \text{ is semidecidable} &\leftrightarrow P \text{ is } (\omega \cdot 2)\text{-decidable} \end{aligned}$$

Closure under conjunction

It is well known that if P and Q are both (semi)decidable, then also $P \wedge Q$ is (semi)decidable.

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Unfortunately, this is not possible:

Proposition: Assume that, for all ordinals, their minimum exists. Then, LPO holds.

Salvaging closure under conjunction

Thankfully, we do not need to compute minima in general; it suffices to do so for limit ordinals.

Definition: Define $\mathsf{limMin} : \mathsf{Brw} \to \mathsf{Brw} \to \mathsf{Brw}$ by

$$\begin{aligned}
\mathsf{limMin}\,\mathsf{zero}\,\beta &= \mathsf{zero} \\
\mathsf{limMin}\,(\mathsf{succ}\,\alpha)\,\beta &= \mathsf{limMin}\,\alpha\,\beta \\
\mathsf{limMin}\,(\mathsf{limit}\,f)\,\mathsf{zero} &= \mathsf{zero} \\
\mathsf{limMin}\,(\mathsf{limit}\,f)\,(\mathsf{succ}\,\beta) &= \mathsf{limMin}\,(\mathsf{limit}\,f)\,\beta \\
\mathsf{limMin}\,(\mathsf{limit}\ f)\,(\mathsf{limit}\ g) &= \mathsf{limit}\,(\lambda\,n.\,(\mathsf{limMin}\,f_n\,g_n) + n)
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(In practice, we define the graph of limMin as an inductive relation, and prove it functional.)

Lemma: For zero or successors $x, y : \mathsf{Brw}^{zl}$ and $\alpha : \mathsf{Brw}$, we have

$$\big(\alpha \leq x \wedge \alpha \leq y\big) \longleftrightarrow \alpha \leq \mathsf{limMin}\,x\,y.$$

Proof of closure under conjunction

Theorem: If both P and Q are α-decidable, then also $P \wedge Q$ is α-decidable.

Proof: Assume we are given x and y with $P \leftrightarrow \alpha \leq x$ and $Q \leftrightarrow \alpha \leq y$.

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Hence $P \wedge Q \leftrightarrow \big(\alpha \leq x \wedge \alpha \leq y\big) \leftrightarrow \alpha \leq \mathsf{limMin}\, x\, y$. □

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Intuitive reason: the existence of maxima even restricted to Brw^{zl} is not constructively possible:

Proposition: Assume that there exists $m : \mathsf{Brw}^{zl} \to \mathsf{Brw}^{zl} \to \mathsf{Brw}^{zl}$ such that, for all $\alpha, x, y : \mathsf{Brw}^{zl}$, we have

$$\alpha \leq m(x, y) \longleftrightarrow \big(\alpha \leq x \vee \alpha \leq y\big).$$

Then, $x \leq y \vee y \leq x$ holds, which implies LPO.

Maxima of limits for small α

Nonetheless, we can compute maxima with partial success:

Definition: Define $\mathsf{limMax} : \mathsf{Brw} \to \mathsf{Brw} \to \mathsf{Brw}$ by

$$\begin{aligned}
\mathsf{limMax}\,\mathsf{zero}\,\beta &= \beta \\
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Lemma: For all α, x, y : Brw^{zl}, we have

$$\big(\alpha \leq x \vee \alpha \leq y\big) \rightarrow \alpha \leq \mathsf{limMax}\,x\,y.$$

If $\alpha = \omega \cdot k + n$ for some $k, n : \mathbb{N}$, we also have

$$\alpha \leq \mathsf{limMax}\,x\,y \rightarrow \big(\alpha \leq x \vee \alpha \leq y\big).$$

Closure under disjunction for α below ω^2

All in all, we have:

Theorem: Let $\alpha = \omega \cdot k + n$. If both P and Q are α-decidable, then also $P \vee Q$ is α-decidable.

This generalises the result that decidable ($\alpha = \omega$) and semidecidable ($\alpha = \omega \cdot 2$) propositions are closed under disjunction.

Closure under $\forall$

Theorem: Let $P : \mathbb{N} \to \mathsf{Prop}$. If each $P(n)$ is semidecidable (i.e., $(\omega + 1)$-decidable), then $\forall n.P(n)$ is ω^2-decidable.

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Remark: We do not assume countable choice, so we have to take care that our construction does not depend on the particular witnesses of semidecidability.

Closure under $\exists$

Theorem: Let $P : \mathbb{N} \to \mathsf{Prop}$. If each $P(n)$ is semidecidable (i.e., $(\omega \cdot 2)$-decidable), then $\exists n.P(n)$ is $(\omega \cdot 3)$-decidable.

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Theorem: If $P : \mathbb{N} \times \mathbb{N} \to \mathsf{Prop}$ is a family of semidecidable propositions such that for all m and n

$$P(n, m) \to P(n, m + 1),$$

then $\exists m. \forall n. P(n, m)$ is $(\omega^2 + \omega)$-decidable.

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In particular, the search for a counterexample to the Twin Prime Conjecture is $(\omega^2 + \omega)$-decidable. (Take

$$P(n, m) = \text{if } n > m \text{ then } n \text{ or } n + 2 \text{ is not prime.})$$

Consequences of countable choice

Traditional computability theory usually assumes countable choice. For example:

Lemma: Assuming countable choice, semidecidable propositions are closed under countable joins (i.e., under "$\exists n : \mathbb{N}.$").

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From our point of view, this says that certain $(\omega \cdot 3)$-decidable propositions are $(\omega \cdot 2)$-decidable. We can do slightly better:

Theorem: Assuming countable choice, every $(\omega \cdot k)$-decidable proposition is $(\omega \cdot 2)$-decidable (i.e., semidecidable).

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Theorem: Assume every ω^2-decidable proposition is $(\omega \cdot 2)$-decidable. Then, Markov's Principle implies LPO.

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Remark: There are models where Markov's Principle and Countable Choice holds, but LPO does not [Hendtlass and Lubarsky, 2016]. Hence countable choice cannot collapse the whole hierarchy, even if it collapses $(\omega \cdot k)$-decidability to $(\omega \cdot 2)$-decidability.

Open questions

Upwards closure: If P is α-decidable and $\alpha \leq \beta$, is P also β-decidable? We think this should at least hold for $\alpha, \beta < \varepsilon_0$.

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The arithmetical hierarchy: If $P(n)$ is α-decidable for each $n : \mathbb{N}$, what can in general be said about $\forall n. P(n)$ and $\exists n. P(n)$, when each $P(n)$ is not assumed to be semidecidable?

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Other notions of ordinals: Are Brouwer trees the best ordinals to use to categorise decidability? What about Brouwer trees with limits of not just $\mathbb{N}$-indexed sequences?

Summary

We have introduced the notion of α-decidability, for a Brouwer tree α.

This generalizes existing notions of decidability and semidecidabilty.

The α-decidable propositions are closed under conjunction, and many other connectives and quantifiers for more restricted α.

📄 Generalized Decidability via Brouwer Trees.
Tom de Jong, Nicolai Kraus, Fredrik Nordvall Forsberg and Aref Mohammadzadeh. Will appear on the arXiv soon.

Full Agda formalisation.
Building on the agda/cubical library.

`https://bitbucket.org/nicolaikraus/constructive-ordinals-in-hott/`

References

- Andrej Bauer. 2006. "First steps in synthetic computability theory", ENTCS, volume 155. doi:10.1016/j.entcs.2005.11.049.
- Giuseppe Rosolini. 1986. "Continuity and Effectiveness in Topoi", PhD thesis, University of Oxford.
- Nicolai Kraus, Fredrik Nordvall Forsberg, and Chuangjie Xu. 2023. "Type-theoretic approaches to ordinals". Theoretical Computer Science, 957:113843, 2023. doi:10.1016/j.tcs.2023.113843.
- Matt Hendtlass and Robert Lubarsky. 2016. "Separating fragments of WLEM, LPO, and MP". The Journal of Symbolic Logic, 81(4):1315–1343, 2016. doi:10.1017/jsl.2016.38..
- The agda/cubical development team. 2018–. The agda/cubical library. Available at: `https://github.com/agda/cubical/`.