Constructive taboos for ordinals

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joint work with **Nicolai Kraus** and **Chuangjie Xu**

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Ordinals

What is an ordinal number?

One answer: The essence of counting beyond the finite.

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Another answer: The essence of termination.

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Set theory answer: a transitive, wellfounded and extensional order (cf. Taylor [1996]).

Transitive, wellfounded and extensional orders

The Homotopy Type Theory Book defines the type Ord as the type of sets equipped with an order $\prec$, which is

- **transitive**: $(a \prec b) \to (b \prec c) \to (a \prec c)$
- **wellfounded**: transfinite induction along $\prec$ is valid
- and **extensional**: $(\forall a.a \prec b \leftrightarrow a \prec c) \to b = c$

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Theorem (Escardo [2022])

The type Ord has a non-trivial decidable property if and only if weak excluded middle $\neg P \uplus \neg\neg P$ holds.

This motivates a search for representations of ordinals that can be more useful constructively.

What has the ordinals ever done for us?

Two typical uses of ordinals:

- Transfinite iteration of operators
- Termination of processes

Transfinite iteration

Let F : Set $\rightarrow$ Set be a finitary functor.

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The initial algebra of F can be constructed as the colimit of the sequence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots$$

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Useful: Definitional principle where ordinals are classified as 0, $\alpha + 1$ or a limit.

Termination of processes

- Programs terminating [Turing 1949]
- Consistency proof e.g. of Peano's axioms [Gentzen 1936]
- Termination of Goodstein sequences [Goodstein 1944], the Hydra game [Kirby&Paris 1982]:

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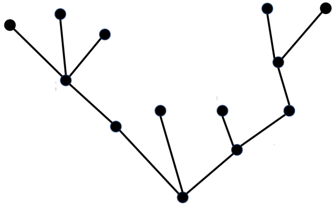

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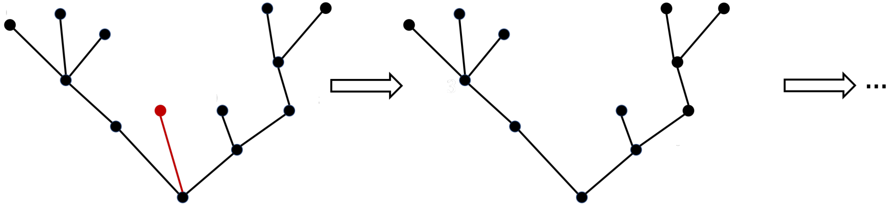

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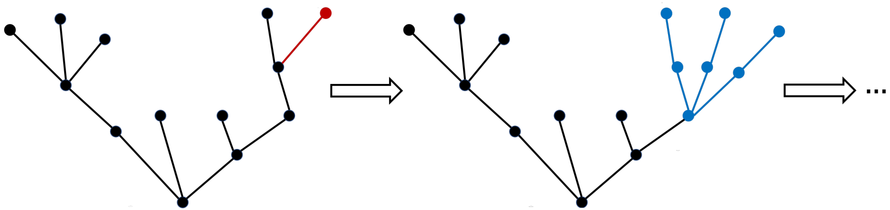

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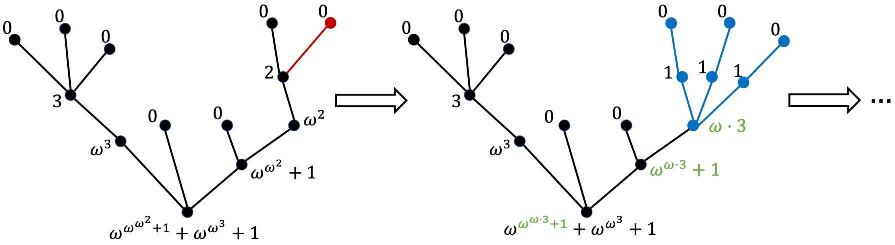

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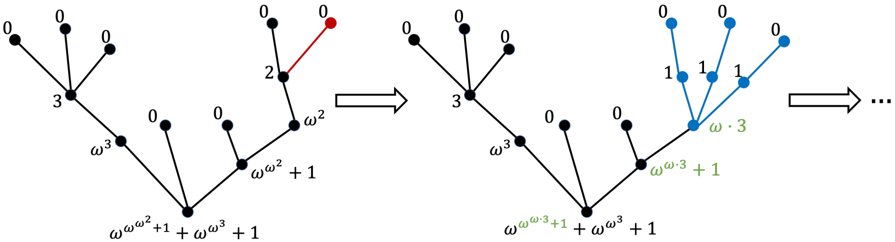

Useful: Arithmetic, and every decreasing sequence of ordinals hits 0.

A spectrum of ordinal notions

decidable	Cantor normal forms
	$\big\downarrow$ (hooked arrow)
partially decidable	Brouwer trees
	$\big\downarrow$ (hooked arrow)
undecidable	Wellfounded, extensional, and transitive orders

N. Kraus, F. N-F., and C. Xu.
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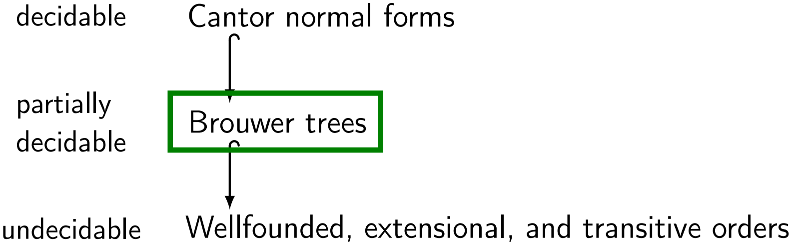

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Brouwer ordinal trees in constructive type theory

Inductive type $\mathcal{B}$ of Brouwer trees [Brouwer 1926; Martin-Löf 1970]:

data $\mathcal{B}$ where

zero : $\mathcal{B}$

succ : $\mathcal{B} \to \mathcal{B}$

limit : $(\mathbb{N} \to \mathcal{B}) \to \mathcal{B}$

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Examples:

$\omega :=$ limit$(0, 1, 2, 3, \ldots)$

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and so on (addition, multiplication, exponentiation are standard).

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A refined type of Brouwer tree ordinals

```
data Brw where
  zero  : Brw
  succ  : Brw → Brw
  limit : (f : ℕ → Brw) → {f↑ : increasing f} → Brw
  bisim : ∀ f {f↑} g {g↑} →
          f ≈ g →
          limit f {f↑} ≡ limit g {g↑}
  trunc : isSet Brw

data _≤_ where
  ≤-zero      : ∀ {x} → zero ≤ x
  ≤-trans     : ∀ {x y z} → x ≤ y → y ≤ z → x ≤ z
  ≤-succ-mono : ∀ {x y} → x ≤ y → succ x ≤ succ y
  ≤-cocone    : ∀ {x} f {f↑ k} → (x ≤ f k) → (x ≤ limit f {f↑})
  ≤-limiting  : ∀ f {f↑ x} → ((k : ℕ) → f k ≤ x) → limit f {f↑} ≤ x
  ≤-trunc     : ∀ {x y} → isProp (x ≤ y)
```

note: $x < y$ means succ $x \leq y$

$f \approx g$ means $\forall k . \exists n . f(k) \leq g(n)$ and vice versa

- Induction-induction (N.-F. [2013]): limits can only be taken of increasing sequences;
- Path constructor (Lumsdaine and Shulman [2020]): bisimilar sequences have equal limits.

Recursion and induction principles for Brw

To define $f : \mathsf{Brw} \to X$ for $X : \mathsf{Set}$, it suffices to give

$$f\,\mathsf{zero} = ?_0$$

$$f\,(\mathsf{succ}\,x) = ?_1 \qquad (\text{given } f\,x)$$

$$f\,(\mathsf{limit}\,g) = ?_2 \qquad (\text{given } f\,(g\,i) \text{ for any } i : \mathbb{N})$$

such that $f\,(\mathsf{limit}\,g) = f\,(\mathsf{limit}\,h)$ whenever $g \approx h$.

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such that $f\,(\mathsf{limit}\,g) = f\,(\mathsf{limit}\,h)$ whenever $g \approx h$.

To prove $\forall (x : \mathsf{Brw}).\, P(x)$ for $P : \mathsf{Brw} \to \mathsf{Prop}$, it suffices to give

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\begin{aligned}
&p_{\mathsf{zero}} : P\,\mathsf{zero} \\
&p_{\mathsf{succ}}\,x : P\,x \to P\,(\mathsf{succ}\,x) \\
&p_{\mathsf{limit}}\,g : (\forall (i : \mathbb{N}).\, P\,(g\,i)) \to P\,(\mathsf{limit}\,g)
\end{aligned}
$$

(Note $p_{\mathsf{limit}}\,g = p_{\mathsf{limit}}\,h$ for $g \approx h$ follows always, since P is Prop-valued.)

Example: multiplication

Seemingly straightforward definition:

$$
\begin{aligned}
x \cdot \mathsf{zero} &= \mathsf{zero} \\
x \cdot (\mathsf{succ}\, y) &= x \cdot y + x \\
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$$x \cdot (\mathsf{limit}\ f\ \{\textit{incr-f}\})\ \mathsf{with}\ \mathsf{decZero}\ x$$
$$\ldots \mid \mathsf{yes}\ \textit{x}{\equiv}\textit{0} = \mathsf{zero}$$
$$\ldots \mid \mathsf{no}\ \textit{x}{\not\equiv}\textit{0} = \mathsf{limit}\ (\lambda i.\, x \cdot f_i)\ \{\mathsf{x{\cdot}\text{-}increasing}\ \textit{x}{\not\equiv}\textit{0}\ \textit{incr-f}\}$$

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Basic feasibility

Everything that one can “reasonably expect” works:

- $<$ is wellfounded and extensional;
- $\leq$ is antisymmetric;
- limits are actually limits;
- $\mathsf{zero} \neq \mathsf{succ}\,x$, $\mathsf{succ}\,x \neq \mathsf{limit}\,g$, etc;
- arithmetic operations can be defined and proven correct;
- and so on.

Characterising $\leq$ using encode-decode

Main proof technique: we use an encode-decode method [Licata and Shulman 2013] to characterise the $\leq$ relation.

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Technically involved: need to simultaneously prove transitivity, reflexivity of Code, and $(x \leq y) \to \mathsf{Code}\, x\, y$.

Decidability properties

P is *decidable* if we can prove $\mathsf{Dec}\,P :\equiv P \uplus \neg P$.

data Brw where

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2. $x = 5$?

   Sure: No for zero and limits; for succ y, check whether $y = 4$.

Decidability properties

data Brw where

zero : Brw

succ : Brw $\to$ Brw

limit : $(\mathbb{N} \xrightarrow{incr}$ Brw$) \to$ Brw

P is *decidable* if we can prove Dec $P :\equiv P \uplus \neg P$.

If x is a Brouwer tree ordinal, is it decidable whether ...

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   Sure: zero is finite; succ y is finite iff y is; limits are never finite.

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4. $x > \omega$?
   Can decide it for zero and succ, but: limit$(x_0, x_1, x_2, \ldots) > \omega$?

When is limit$(x_0, x_1, x_2, \ldots) > \omega$?

- For any i, we can check whether x_i is finite.
- As soon as we discover an infinite x_i, the question is decided positively.
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Indeed if we assume the *lesser principle of omniscience*

$$\text{LPO} :\equiv \forall (s : \mathbb{N} \to \text{Bool}).(\forall n. s_n = \text{false}) \uplus (\exists n. s_n = \text{true}).$$

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the question $x > \omega$ is decidable. Conversely:

Theorem

$$(\forall x : \mathit{Brw}. \mathsf{Dec}\,(x > \omega)) \leftrightarrow \mathsf{LPO}$$

$\forall x : \mathsf{Brw.Dec}\,(x > \omega)$ implies LPO

Given $s : \mathbb{N} \to \mathsf{Bool}$, we can construct an increasing sequence $s^{\uparrow} : \mathbb{N} \to \mathsf{Brw}$ by

$$s^{\uparrow}\, n = \begin{cases} \omega + n & \textit{if there is } k \leq n \textit{ such that } s_k = \textit{true} \\ n & \textit{else.} \end{cases}$$

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Then: $(\mathsf{limit}\, s^{\uparrow} > \omega) \leftrightarrow (\exists k. s_k = \mathsf{true})$.

Key lemma: If $y < \mathsf{limit}\, f$, then $\exists k. y < f\, k$.

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Then: $(\mathsf{limit}\, s^{\uparrow} > \omega) \leftrightarrow (\exists k . s_k = \mathsf{true})$.

Key lemma: If $y < \mathsf{limit}\, f$, then $\exists k . y < f\, k$.

Hence if we can decide $\mathsf{limit}\, s^{\uparrow} > \omega$, we know whether $\forall n . s_n = \mathsf{false}$ or $\exists n . s_n = \mathsf{true}$.

Many decidability statements for Brw are equivalent to LPO

Using similar proof ideas, we can show:

Theorem

For the type of Brouwer trees, the following statements are equivalent:

(i) LPO

(ii) $\forall x, y.\mathsf{Dec}(x \leq y)$

(iii) $\forall x, y.\mathsf{Dec}(x < y)$

(iv) $\forall x, y.\mathsf{Dec}(x = y)$

(v) $\forall x.\mathsf{Dec}(\omega < x)$

(vi) $\forall x.\mathsf{Dec}(x = \omega \cdot 2)$

A slight generalisation

Lemma

For α, β : *Brw and* $k : \mathbb{N}$, *we have*

(i) $(\forall x.\mathsf{Dec}(x = \beta + \alpha)) \rightarrow (\forall x.\mathsf{Dec}(x = \alpha))$

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Proof sketch.

For (i), note that addition is left cancellative:

$$\beta + x = \beta + \alpha \to x = \alpha$$

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Proof sketch.

For (i), note that addition is left cancellative:

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For (ii), we can decide if x starts with k successors or not. □

Equality with $\omega \cdot n + k$

Theorem

Let x : Brw. We have:

$$\mathrm{Dec}(x = k) \leftrightarrow \textit{True}$$

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Let x : Brw. We have:

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Equality with $\omega \cdot n + k$

Theorem

Let x : Brw. We have:

$$\begin{aligned} \mathsf{Dec}(x = k) &\leftrightarrow \textit{True} \\ \mathsf{Dec}(x = \omega + k) &\leftrightarrow \mathsf{WLPO} \\ \mathsf{Dec}(x = \omega \cdot (n+2) + k) &\leftrightarrow \mathsf{LPO} \end{aligned}$$

$$\mathsf{WLPO} :\equiv \forall (s : \mathbb{N} \to \mathsf{Bool}).(\forall n. s_n = \mathsf{false}) \uplus \neg(\forall n. s_n = \mathsf{false})$$

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P is $\neg\neg$-*stable* if we can prove $\mathsf{Stable}\, P :\equiv (\neg\neg P \to P)$.

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P is ¬¬-*stable* if we can prove Stable $P :\equiv (\neg\neg P \to P)$.

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Trichotomy

Classically, ordinals satisfy $(x < y) \uplus (x = y) \uplus (x > y)$.

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Theorem

For the type of Brouwer trees, the following are equivalent:

(i) LPO

(ii) *trichotomy:* $\forall x, y.(x < y) \uplus (x = y) \uplus (y < x)$

(iii) *splitting:* $\forall x, y.(x \leq y) \rightarrow (x < y) \uplus (x = y)$.

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Proof sketch.

(i) $\Rightarrow$ (ii): LPO implies $\neg(x < y) \to y \leq x$. Use LPO to decide $x < y$ and $y < x$.

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(ii) $\Rightarrow$ (iii): We cannot have both $y < x$ and $x \leq y$ by irreflexivity.
(iii) $\Rightarrow$ (i): We always have $s^{\uparrow} \leq \omega \cdot 2$. Further $s^{\uparrow} = \omega \cdot 2 \leftrightarrow \exists k.s_k =$ true. □

Taboo arithmetic

The usual ordinal arithmetic operations can be defined for all notions of ordinals we consider, and proven correct.

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Definition

A notion of ordinals A has *subtraction*, if there is an operation $(b : A) \to (a : A) \to (p : a \leq b) \to A$, written $b -_p a$, such that $a + (b -_p a) = b$.

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Perhaps surprisingly, having subtraction is a constructive taboo for Brw:

Theorem

Brw has subtraction if and only if LPO *holds.*

Subtraction is a taboo

Theorem

Brw has subtraction if and only if $\leq$ *splits, i.e.* $(x \leq y) \to (x < y) \uplus (x = y)$*.*

Proof sketch.

$\square$

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Theorem

Brw has subtraction if and only if $\leq$ splits, i.e. $(x \leq y) \to (x < y) \uplus (x = y)$.

Proof sketch.

If Brw has subtraction and $p : x \leq y$, then $x = y$ iff $y -_p x = 0$, which is always decidable.

□

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Conversely, note that "having subtraction" is a proposition by left cancellation:

$$x + (y -_p x) = y = x + (y -_p x)' \quad \text{so } (y -_p x) = (y -_p x)'$$

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Hence we can define $y -_p x$ by induction on y. Splitting p, we define $y -_p y = 0$, and if $x < y$, we can use the induction hypothesis to finish the definition. □

Binary joins

We only compute limits of increasing sequences limit $(s_0, s_1, s_2, \ldots)$. What if we relaxed this requirement?

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Theorem

If $y = n$ for a finite n, or $y = \omega$, we can define a function $(- \sqcup y) :$ Brw $\to$ Brw calculating the binary join with y.

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Theorem

If $y = n$ for a finite n, or $y = \omega$, we can define a function $(- \sqcup y)$: Brw $\rightarrow$ Brw calculating the binary join with y.

However this is as far as we can go; already computing $x \sqcup (\omega + 1)$ is a constructive taboo.

Theorem

LPO *implies* $(- \sqcup (\omega + 1))$ *can be calculated, which in turn implies* WLPO*.*

Beyond decidability

Semidecidability via Brouwer trees

Definition (Bauer [2006], cf. also Veltri [2017])

P is *semidecidable* if $\exists(s : \mathbb{N} \to \mathsf{Bool})\big(P \leftrightarrow \exists k. s_k = \mathsf{true}\big)$.

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Recall construction of $s^{\uparrow}$ with $\mathsf{limit}\, s^{\uparrow} > \omega \leftrightarrow \exists k.s_k = \mathsf{true}$.

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Fact: For any proposition P,

$$\exists(y : \mathsf{Brw})\big(P \leftrightarrow (y > \omega)\big) \qquad \longleftrightarrow \qquad \exists(s : \mathbb{N} \to \mathsf{Bool})\big(P \leftrightarrow \exists k.s_k = \mathsf{true}\big)$$

"P decidable in ω steps" — "P semidecidable"

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"P decidable in ω steps" — "P semidecidable"

What if we swap ω for another ordinal α?

Definition

P is *decidable in α steps* if $\exists(y : \mathsf{Brw})\big(P \leftrightarrow (y > \alpha)\big)$.

Fewer than ω steps

Theorem

Let n be a natural number. Then:

$$\exists(y : \textit{Brw})\big(P \leftrightarrow (y > n)\big) \qquad \longleftrightarrow \qquad P \uplus \neg P$$

"P decidable in n steps" — *"P decidable"*

More than ω steps – an example

Twin prime conjecture (TPC):

There are arbitrarily large numbers p such that p and $p + 2$ are both prime.

It is clearly semidecidable whether there is a twin pair $> 10^{1,000,000}$, but TPC does not seem to be semidecidable.

More than ω steps – an example

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It is clearly semidecidable whether there is a twin pair $> 10^{1,000,000}$, but TPC does not seem to be semidecidable.

However, one can show:

$$\exists(y : \mathsf{Brw})\big(\mathsf{TPC} \leftrightarrow (y > \omega^2)\big)$$

"TPC is decidable in ω^2 steps."

TPC's ordinal

Define a sequence $f : \mathbb{N} \to \mathsf{Brw}$ by:

$$f\,0 = \mathsf{zero}$$

$$f\,(n+1) = \begin{cases} (f\,n) + \omega & \textit{if } n \textit{ and } n+2 \textit{ are prime} \\ (f\,n) + 1 & \textit{else.} \end{cases}$$

Claim

$\big(\forall n.\exists p > n.\ p,\ p+2 \textit{ are prime}\big) \ \leftrightarrow\ \textit{limit}\, f = \omega^2 \ \leftrightarrow\ \textit{succ}\,(\textit{limit}\, f) > \omega^2$

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Proof sketch TPC $\to$ (limit $f = \omega^2$).

For any n, we find $p > n$ s.t. $f(p) \geq \omega \cdot p$, thus limit $f \geq \omega \cdot \omega$.
At the same time, f never exceeds ω^2. $\square$

TPC's ordinal

Define a sequence $f : \mathbb{N} \to \mathsf{Brw}$ by:

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Proof sketch $(\text{limit}\,f \geq \omega^2) \to$ TPC.

For every n, $(\text{limit}\,f \geq \omega^2) \quad \Rightarrow \quad \exists k. f_k \geq \omega \cdot (n+1)$

$\Rightarrow \quad \exists k. \neg\neg(\ f(p)$ jumped for some $n < p \leq k)$

$\Rightarrow \quad \exists k.\ f(p)$ jumped for some $n < p \leq k$

$\Rightarrow$ there is a twin prime pair $(p, p+2)$ above n

Summary

We have considered decidability aspects of different notions of ordinals.

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Cantor normal forms

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"Decidability ↔ True"

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Type-Theoretic Approaches to Ordinals
arXiv:2208.03844

Summary

We have considered decidability aspects of different notions of ordinals.

"Decidability ↔ True" — Cantor normal forms

↓

Brouwer trees

↓

"Decidability ↔ (W)LEM" — Wellfounded, extensional, and transitive orders

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"Decidability $\leftrightarrow$ True"	Cantor normal forms
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"Finite decidability $\leftrightarrow$ True" "Infinite decidability $\leftrightarrow$ (W)LPO"	Brouwer trees
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Thank you!

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